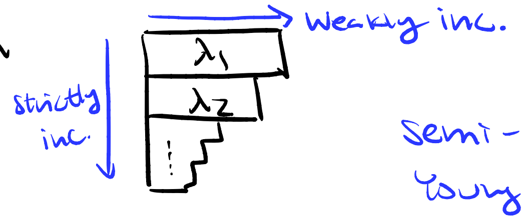
Symmetric Functions and Young Tableaux Pt 2

Recall: A partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ of n can be identified with a Young Diagram

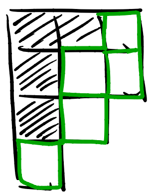


Semi-standard Young Tableaux (SSYT)

Def: If $\mu \subseteq \lambda$ as Young Diagrams then the corresponding skew shape (diagram) is $\lambda/\mu := \{\text{boxes } b \mid b \in \lambda \text{ and } b \notin \mu\}$.

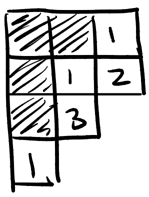
ex: $\lambda = (3,3,2,1)$, $\mu = (2,1,1)$

$\lambda/\mu =$



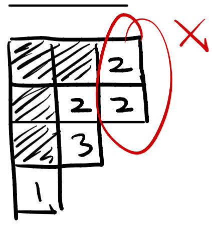
Def: A SSYT of shape λ/μ is a filling of λ/μ such that the rows are weakly increasing and the columns are strictly increasing.

ex: λ, μ as above.



✓

non ex:



Recall: Schur polynomials

For $\lambda \vdash n$,

$$s_\lambda(x_1, \dots, x_n) = \sum_{T \in SSYT(\lambda)} x^{wt(T)} = \sum_{T \in SSYT(\lambda)} x_1^{\#\text{of 1's}} \cdots x_n^{\#\text{of } n\text{'s}}$$

Skew Schur polynomials defined similarly

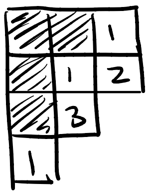
Since the S_λ's form a basis for the ring of symmetric polynomials, we want to understand the structure constants

$$S_\lambda \cdot S_\mu = \sum_{\nu} \underline{C^{\nu}_{\lambda,\mu}} \, S_\nu$$

Theorem (Littlewood-Richarson Rule) (1934)

$$C^{\nu}_{\lambda\mu} = \# \left\{ \begin{array}{l} T \in SSYT(\nu/\lambda) \text{ weight } \mu \text{ such that} \\ \text{the reading word of } T \text{ is reverse lattice} \end{array} \right\}$$

"Littlewood-Richardson Coefficients"



reading word = read entries bottom to top left to right.

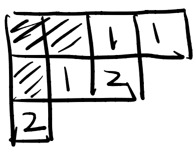
1 3 1 2 1

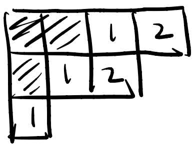
reverse lattice: each subword reading right to left has at least as many i's as $i+1$'s

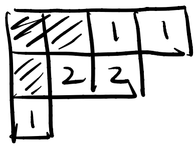
13121 is reverse lattice ✓

Example $\nu = (4,3,1)$ $\lambda = (2,1)$, $\mu = (3,2)$

$SSYT(\nu/\lambda, \mu)$:







reading word : 21211 11212 12211

✓ reverse lattice ✗ ✓

$$\Rightarrow \quad C^{\nu}_{\lambda,\mu} = 2$$

History of Littlewood-Richardson Rule

1934: Littlewood & Richardson. wrong proof and example

1938: Robinson. wrong proof.

1974 or 78? Thomas proves it through RSK

1977: Schützenberger proves it through jeu de taquin

⋮ (many more proofs)

Wikipedia

The Littlewood–Richardson rule is notorious for the number of errors that appeared prior to its complete, published proof. Several published attempts to prove it are incomplete, and it is particularly difficult to avoid errors when doing hand calculations with it: even the original example in D. E. Littlewood and A. R. Richardson (1934) contains an error.

> suspected. The author was once told that the Littlewood–Richardson rule helped to get men on the moon but was not proved until after they got there.
>
> Gordon James (1987)

IGOR PAK*

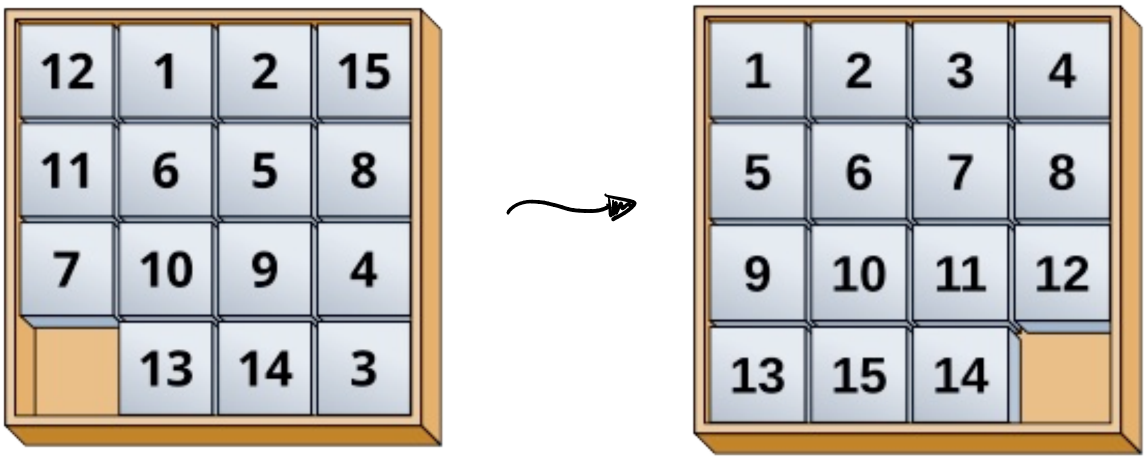
11.4. **LR rule.** Consider the *Littlewood–Richardson coefficients* and its many combinatorial interpretations (see e.g. [vL01] for an extensive albeit dated survey):

- The *original LR rule*: $c^{\lambda}_{\mu\nu} = |\,\mathrm{LR}(\lambda/\mu,\nu)|$, see [LR34].
- The *LR variation*: $c^{\lambda}_{\mu\nu} = |\,\mathrm{LR}(\mu\circ\nu,\lambda)|$, see e.g. [RW84].
- James–Peel *pictures* [JP79], see also [CS84, Zel81].
- Gelfand–Zelevinsky interpretation using *Gelfand–Tsetlin patterns* [GZ85].
- Leaves of the *Lascoux–Schützenberger tree* [LS85].
- Kirillov–Reshetikhin *rigged configurations* [KR88] (see also [KSS02]).
- Berenstein–Zelevinsky *triangles* [BZ92].
- Fomin–Greene *good maps* [FG93].
- Nakashima's interpretation using *crystal graphs* [Nak93] (see also [BS17, §9]).
- Littelmann's *paths* [Lit94].
- Knutson–Tao *hives* [KT99], see also [GP00].
- Kogan's interpretation using *RC-graphs* [Kog01].
- Buch's *set-valued tableaux* [Buch02].
- Knutson–Tao–Woodward *puzzles* [KTW04].
- Danilov–Koshevoy *arrays* [DK05a].
- Vakil's *chessgames* [Vak06].
- Thomas–Yong *S_3-symmetric LR rule* [TY08].
- Purbhoo's *mosaics* [Pur08] (see also [Zin09]).
- Coskun's *Mondrian tableaux* [Cos09].
- Nadeau's *fully packed loop configurations in a triangle* [Nad13] (see also [FN15]).

The list above is so lengthy, it is worth examining carefully. Most of these LR rules are byproducts of (often but not always, successful) efforts to find a combinatorial interpretation of more general numbers. Some of these are closely related to each other, while others seem quite different, both visually and mathematically.

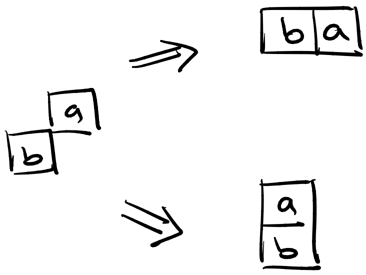
Jeu de taquin

("teasing game"
French name for the 15 puzzle)



Goal: Turn a skew shape SSYT into a SSYT by "sliding" boxes.

Rule:



if $b \le a$

if $a < b$

Definition

empty box with entries to the right & below.

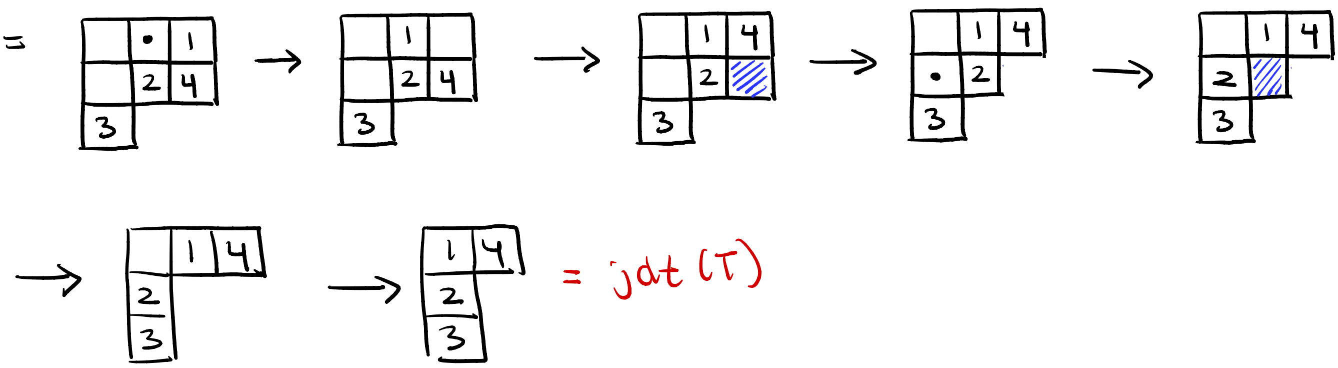
Let x be an inner corner of $T \in SSYT(\lambda/\mu)$

A jeu de taquin (jdt) slide for x is a tableau T' of shape $\lambda - \{\text{some outer corner}\} / \mu - \{x\}$ obtained by sliding x until it becomes an outer corner.

no box to the right or below.

ex

$T =$



$= \mathrm{jdt}(T)$

Theorem Fix a standard Young tableau P of shape μ

$$c^{\nu}_{\lambda,\mu} = \#\{ T \in SYT(\nu/\lambda) \mid jdt(T) = P \}$$

Example $\nu = (6,4,2,1)$ $\lambda = (3,2)$, $\mu = (4,3,1)$

$$P = \begin{array}{|c|c|c|c|} \hline 1 & 3 & 5 & 8 \\ \hline 2 & 4 & 6 & \\ \cline{1-3} 7 & & & \\ \cline{1-1} \end{array}$$

$$T_1 = \begin{array}{|c|c|c|c|c|c|} \hline \ & \ & \ & 3 & 5 & 8 \\ \hline \ & \ & 1 & 6 & & \\ \cline{1-4} 2 & 4 & & & & \\ \cline{1-2} 7 & & & & & \\ \cline{1-1} \end{array} \qquad T_2 = \begin{array}{|c|c|c|c|c|c|} \hline \ & \ & \ & 3 & 5 & 8 \\ \hline \ & \ & 4 & 6 & & \\ \cline{1-4} 1 & 7 & & & & \\ \cline{1-2} 2 & & & & & \\ \cline{1-1} \end{array} \qquad T_3 = \begin{array}{|c|c|c|c|c|c|} \hline \ & \ & \ & 1 & 3 & 5 \\ \hline \ & \ & 6 & 8 & & \\ \cline{1-4} 2 & 4 & & & & \\ \cline{1-2} 7 & & & & & \\ \cline{1-1} \end{array}$$

$$jdt(T_1) = jdt(T_2) = jdt(T_3) = P$$

$$\Rightarrow \quad c^{\nu}_{\lambda,\mu} = 3$$

Representation Theory

- irreducible polynomial representations of $GL_n(\mathbb{C})$ are indexed by partitions $\lambda \vdash n$.

- Denote these irreducible reps by V^λ
- Last time: $\chi(V^\lambda) = s_\lambda$

- $R = \langle [V^\lambda] \mid \lambda \vdash n \rangle$ representation ring of $GL_n(\mathbb{C})$.

  multiplication: $[V]\cdot[W] = V \otimes W$

Theorem: The map $$\begin{aligned} ch : R &\longrightarrow \Lambda[x_1, \dots, x_n] \\ [V^\lambda] &\longmapsto s_\lambda(x_1, \dots, x_n) \end{aligned}$$

is an algebra homomorphism.

$$ch(V \otimes W) = ch(V)\cdot ch(W)$$
$$ch(V \oplus W) = ch(V) + ch(W)$$

Corollary

$$V^\lambda \otimes V^\mu = \bigoplus_\nu (V^\nu)^{\oplus c^\nu_{\lambda,\mu}}$$

i.e. the LR coefficients give the multiplicities of the irreducible reps V^ν appearing in the decomposition of the tensor product of two irreps of GL_n

$$s_\lambda \cdot s_\mu = \sum_\nu c^\nu_{\lambda\mu} s_\nu$$

$$V^\lambda \otimes V^\mu = \bigoplus_\nu (V^\nu)^{\oplus c^\nu_{\lambda,\mu}}$$