Solutions to UC Davis Prelims

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Contents

Introduction

There are likely lots of errors, let me know if you find any.

Analysis

Some tips for problems solving in analysis

(a) Read and do exercises out of Rudin.

(b) Know how to use inequalities.

201 Midterms and Practice Midterms

The following questions are pulled from various 201 midterms and practice midterms which I was able to find on faculty webpages.

Problem (John Hunter Practice MT: Fall 2016). Let A be a subset of a metric space X. Define the characteristic function $\chi_A : X \to \boldsymbol{R}$ of A by

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$$

Prove that A is open if and only if χ_A is lower semi-continuous.

Let A be open and $x_n \to x$, $x \in A$. Then there exists a ball of radius r such that $B_r(x) \subset A$ and for some $N \in \boldsymbol{N}$, $n \geqslant N$ means that $x_n \in B_r(x)$. Then

$$1 = \chi_A(x) = \liminf_{x_n \to x} \chi_A(x_n).$$

Suppose $x \notin A$. Then for any sequence $x_n \to x$

$$\chi_A(x) = 0 \leqslant \liminf_{x_n \to x} \chi_A(x_n)$$

and χ_A is lower semincontinuous.

Conversely suppose that χ_A is lower semicontinuous and suppose for the sake of a contradiction that A is not open. Then there exists some $x \in A$ such that for every $r > 0$, $B_r(x) \cap X \setminus A \neq \varnothing$. Choose a sequence $x_n \to x$ such that each $x_n \in X \setminus A$. Then

$$\chi_A(x) = 1 \geqslant 0 = \liminf_{x_n \to x} \chi_A(x_n)$$

which is a contradiction since χ_A is lower semicontinuous. Therefore A must be open as desired. ∎

Spring 2020

Problem 1. Let $f \in \mathcal{S}(\boldsymbol{R})$ be a Schwartz function. Suppose $\int_{\boldsymbol{R}} f(y)e^{-y^2}e^{2xy}\,dy = 0$ for all $x \in \boldsymbol{R}$, show that $f \equiv 0$.

We begin by proving a small lemma.

(Convolution Preserves Pariety) Let $f \in \mathcal{S}(\boldsymbol{R})$, be even, and let $K \in L^1(\boldsymbol{R}) \cap C^\infty(\boldsymbol{R})$ be even, then

$$\begin{aligned}(f * K)(x) &= \int_{\boldsymbol{R}} f(y)K(x-y)\,dy = \int_{\boldsymbol{R}} f(-y)K(x-y)\,dy \\ &= \int_{\boldsymbol{R}} f(y)K(x+y)\,dy = \int_{\boldsymbol{R}} f(y)K((-x)-y)\,dy \\ &= (f * K)(-x)\end{aligned}$$

let f be even with the same hypothesis on K, then

$$\begin{aligned}(f * K)(x) &= \int_{\boldsymbol{R}} f(y)K(x-y)\,dy = -\int_{\boldsymbol{R}} f(-y)K(x-y)\,dy \\ &= -\int_{\boldsymbol{R}} f(y)K(x+y)\,dy = -\int_{\boldsymbol{R}} f(y)K((-x)-y)\,dy \\ &= -(f * K)(-x)\end{aligned}$$

as desired. □

Next recall the following facts

(i)

$$\int_{\boldsymbol{R}} e^{-x^2}\,dx = \sqrt{\pi}$$

(ii) For $f \in \mathcal{S}(\boldsymbol{R})$, the Fourier inversion formula says

$$f(x) = \int_{\boldsymbol{R}} \hat{f}(\xi)e^{-2\pi i x\xi}\,d\xi$$

(iii)

$$\widehat{(f * g)}(\xi) = \hat{f}(\xi)\hat{g}(\xi)$$

(iv)

$$(e^{-\pi^2\xi^2})^\vee = \frac{1}{\sqrt{\pi}}e^{-x^2}$$

We are now ready to solve our problem. We have

$$\begin{aligned}
\int_{\boldsymbol{R}} f(y)e^{2xy}e^{-y^2}\,dy &= \int_{\boldsymbol{R}}\int_{\boldsymbol{R}} \hat{f}(\xi)e^{2\pi iy\xi+2xy-y^2}\,d\xi\,dy && \text{(ii)}\\
&= \int_{\boldsymbol{R}} \hat{f}(\xi)\int_{\boldsymbol{R}} e^{2\pi iy\xi+2xy-y^2}\,dy\,d\xi && \text{(Fubini)}\\
&= e^{x^2}\int_{\boldsymbol{R}} \hat{f}(\xi)e^{-\pi^2\xi^2}e^{2\pi ix\xi}\int_{\boldsymbol{R}} e^{-(y-(\pi i\xi+x))^2}\,dy\,d\xi && \text{(Complete Square)}\\
&= \sqrt{\pi}e^{x^2}\int_{\boldsymbol{R}} \hat{f}(\xi)e^{-\pi^2\xi^2}e^{2\pi ix\xi}\,d\xi && \text{(i)}\\
&= \sqrt{\pi}e^{x^2}(f * \frac{1}{\sqrt{\pi}}e^{-x^2})(x) = 0 && \text{(iii)}
\end{aligned}$$

which implies that $(f * e^{-x^2})(x) = 0$. By (iv) we see that f is both even and odd, and hence $f = 0$. ■

Problem 4. Suppose that $f \in L^2(\boldsymbol{R})$ and $\hat{f}$ is continuous. Suppose that $\hat{f} = \mathcal{O}(|\xi|^{-1-\alpha})$ as $|\xi| \to \infty$. Show that $|f(x+h) - f(x)| \leqslant Ch^\alpha$ for all $h > 0$ and some constant C independent of h.

We apply the inversion formula

$$\frac{f(x+h)-f(x)}{h^\alpha} = \frac{1}{h^\alpha}\int_{\boldsymbol{R}} \hat{f}(\xi)e^{2\pi ix\xi}(e^{2\pi ih\xi}-1)\,d\xi$$

We estimate

$$\begin{aligned}
\frac{|e^{2\pi ih\xi}-1|}{h^\alpha} &= \left(\frac{|e^{2\pi ih\xi}-1|}{h}\right)^\alpha |e^{2\pi ih\xi}-1|^{1-\alpha}\\
&\leqslant 2^{1-\alpha}\left(\frac{|e^{2\pi ih\xi}-1|}{h}\right)^\alpha
\end{aligned}$$

since $|e^{2\pi ixh}-1| \leqslant 2$

By the mean value theorem we have that

$$|e^{2\pi ih\xi}-1| \leqslant 2\pi|\xi||e^{2\pi i\eta\xi}|h \leqslant 2\pi h|\xi|$$

where $0 < \eta < h$.

Combining the above two estimates shows that

$$\frac{|e^{2\pi ih\xi}-1|}{h^\alpha} \leqslant \min\{2\pi^\alpha|\xi|^\alpha, 2/h^\alpha\}$$

Note that whenever $|\xi| < 1/(\pi h)$, $2\pi^\alpha|\xi|^\alpha < 2$.

Recall that $\hat{f}(\xi) = \mathcal{O}(|\xi|^{-1-\alpha})$ means that there exist constants $M, B \in \boldsymbol{R}^{>0}$ such that whenever $|\xi| > M$, $|\hat{f}(\xi)| \leqslant B|\xi|^{-1-\alpha}$.

Choose $N = \max M, 1/(\pi h)$, then we estimate

$$\begin{aligned}\left|\int_{\boldsymbol{R}} \hat{f}(\xi) e^{2\pi i x\xi} \frac{(e^{2\pi i h\xi}-1)}{h^\alpha} ;\xi\right| &\leqslant \left|\left(\int_{|\xi|<N} + \int_{|\xi|>N}\right) \hat{f}(\xi) e^{2\pi i x\xi} \frac{(e^{2\pi i h\xi}-1)}{h^\alpha}\, d\xi\right| \\ &\leqslant 2\pi^\alpha \int_{|\xi|<N} |\hat{f}(\xi)| |\xi|^\alpha \, d\xi + \frac{2B}{h^\alpha} \int_{|\xi|\geqslant N} |\xi|^{-1-\alpha}\, d\xi \\ &\leqslant 4\pi^\alpha N^\alpha \left(\int_{|\xi|<N} d\xi\right)^{1/2} \left|\left|\hat{f}\right|\right|_{L^2(\boldsymbol{R})} + \frac{4}{h^\alpha} N^{-\alpha} B \\ &= 4\pi^\alpha N (2N)^{1/2} \left|\left|f\right|\right|_{L^1(\boldsymbol{R})} + \frac{4}{h^\alpha} N^{-\alpha} B\end{aligned}$$

By our choice of N we have $N^{-\alpha} \leqslant \pi^\alpha h^\alpha$ so that we have shown

$$\frac{f(x+h)-f(x)}{h^\alpha} \leqslant 2^{5/2} \pi^\alpha M^{3/2} \left|\left|f\right|\right|_{L^2(\boldsymbol{R})} + 4\pi^\alpha B =: C$$

as desired. ■

Problem 6. Find a sequence $\{f_k\}$ of continuous functions on $\boldsymbol{R}$ such that it is uniformly bounded and equi-continuous but fails to have a subsequence that converges uniformly on $\boldsymbol{R}$.

Let

$$f_k(x) = \begin{cases} 0 & , \quad x \leqslant k \\ x-k & , \quad k < x \leqslant k+1 \\ 1 & , \quad k+1 < x \end{cases}$$

it is clear that each f_k is continuous, and it is also easy to see that that $\{f_k\}$ is uniformly bounded by 1, since $|f_k(x)| \leqslant 1$ for all $x \in \boldsymbol{R}$.

We show that $\{f_k\}$ is equi-continuous. Given $\varepsilon > 0$, chosse $0 < \delta < \varepsilon$. For x, y, $|x-y| < \delta$, $x < y$, we have the following cases

- $(x, y) \cap (k, k+1) = \varnothing$.

  In this case $|f_k(x) - f_k(y)| = 0 < \varepsilon$.

- $(x, y) \subset (k, k+1)$

  In this case we have

  $$|f_k(x) - f_k(y)| = |x-k-y+k| = |x-y| < \delta < \varepsilon$$

- $(x, y) \not\subset (k, k+1)$ and $(x, y) \cap (k, k+1) \neq \varnothing$.

  We have two further sub cases. First, suppose $f_k(y) = 1$, then $f_k(x) = x - k$ and $k < y < k+1$ which means

  $$|f_k(x) - f_k(y)| = |x-k-1| < |x-y| < \delta < \varepsilon$$

Next, suppose $f_k(y) = y - k$, then $f_k(x) = 0$ and $k \leqslant y < k+1$ and we have that

$$|f_k(x) - f_k(y)| = |k - y| \leqslant |x - y| < \delta < \varepsilon$$

This proves that $\{f_k\}$ is equicontinuous.

Next note that $f_k(x) \to 0$ as $k \to \infty$ for all $x \in \boldsymbol{R}$ since for any x, $f_k(x) = 0$ for all $k > x$.

Choose any subsequence f_j of $\{f_k\}$ and suppose for the sake of contradiction that f_j converges uniformly. Then for every $\varepsilon > 0$ we have that $|f_j(x)| < \varepsilon$ for all $x \in \boldsymbol{R}$, $j \geqslant J$ for some $J \in \boldsymbol{N}$. However it is easy to see that if we choose $x > j$, $|f_j(x)| = 1$ and hence $|f_j(x)| \not< \varepsilon$ in general. Since $\{f_j\}$ was an arbitrary subsequence, no subsequence converges uniformly. ■

Fall 2019

Problem 1. Let $\mathcal{H}$ be a seperable Hilbert space with orthonormal basis $\{e_k\}_{k\in\boldsymbol{N}}$ and suppose $A \in \mathcal{B}(\mathcal{H})$ is such that

$$\sum_{k=1}^{\infty} ||Ae_k||^2 < \infty$$

Prove A is compact.

An operator A satisfying this condition is known as a Hilbert-Schmidt operator.

We proceed in three parts. We start by showing that the uniform limit of compact operators is compact. Then we show that P_N, the projection onto the N-th dimensional subspace spanned by the first N basis elements is compact. Finally we show that $P_N A \to A$ uniformly in the operator norm topology.

(1) Let X, Y be Banach and let $T_n : X \to Y$ be a sequence of compact operators converging in the operator norm to $T : X \to Y$, that is $||T_n - T|| \to 0$ as $n \to \infty$. Recall that a subset $E \subset Y$ is pre-compact if and only if it is totally bounded.

Let $B \subset X$ be any bounded subset and $\varepsilon > 0$ be given. Let $M > 0$ be such that for all $x \in B$, $||x|| \leqslant M$. Choose $N \in \boldsymbol{N}$ such that $||T_N - T|| < \frac{\varepsilon}{2M}$. Since T_N is bounded, there is a finite indexing family $\mathcal{J}$ of points $t_j \in T_N(B)$ such that $\{t_j\}_{j\in\mathcal{J}}$ is an $\varepsilon/2$-net of $T_N(B)$. Let $x \in B$, then

$$||T_N x - Tx|| = ||(T_N - T)x|| \leqslant ||T_N - T||\,||x|| \leqslant \frac{\varepsilon}{2}$$

so that $Tx \in \boldsymbol{B}_\varepsilon(t_j)$ for some $j \in \mathcal{J}$. Therefore $\{t_j\}_{j\in\mathcal{J}}$ is an ε-net of $T(B)$ and T is compact, as desired.

(2) Let X be a seperable Hilbert space and let $P_N : X \to X$ be the projection operator. The range of P_N is finite dimensional (and hence P_N is finite rank). Note that $P_N(B)$ is bounded since $||P_N x|| \leqslant ||x|| \leqslant M$ for all $x \in B$. Let x_k be a sequence in $P_N(B)$. Since $P_N(B)$ is a finite dimensional vector space and x_k is bounded, x_k has a convergent subsequence. Hence $P_N(B)$ is precompact and P_N is a compact operator.

(3) We have

$$\begin{aligned}
||P_N A - A||^2 &= \sup_{||x||=1} ||P_N Ax - Ax||^2 \\
&= \sup_{||x||=1} \sum_{k>N} |(Ax, e_k)|^2 && \text{(Parseval's)} \\
&\leqslant \sup_{||x||=1} \sum_{k>N} |(Ae_k, e_k)|^2 && \text{(Expanding } x \text{ in basis)} \\
&\leqslant \sup_{||x||=1} \sum_{k>N} ||Ae_k||^2 && \text{(Cauchy Schwartz)} \\
&= \sum_{k>N} ||Ae_k||^2
\end{aligned}$$

so we can see that we may choose N large to make the norm of $P_N A - A$ as small as possible since $\sum_{k\in\boldsymbol{N}} ||Ae_k||^2 < \infty$.

Therefore A is a compact operator. ■

Problem 4. Suppose $f \in C^1([0,1])$ with $f(0) = f(1) = 0$ and $||f||_{L^2} = 1$. Show that

$$||f'(x)||_{L^2} \, ||xf(x)||_{L^2} \geqslant \frac{1}{2}$$

(c.f. Spring 2016 Problem 4: This is the first step towards proving the Heisenburg uncertainty principle)

We compute

$$\begin{aligned} 1 = ||f||_{L^2} &= f^2(1) - f^2(0) - \int_0^1 x\frac{d}{dx}(f^2)\,dx && \text{(IBP)} \\ &= -2\int_0^1 xff'\,dx \\ &\leqslant 2\int_0^1 x|f||f'|\,dx \\ &\leqslant 2\,||xf||_{L^2}\,||f||_{L^2} && \text{(Hölder's)} \end{aligned}$$

which is well defined since $f \in C^1 \implies f \in H^1$. ■

Problem 5. Suppose $f \in L^2(\boldsymbol{R})$. Suppose $\int_R f(y)e^{-y^2}e^{2xy}\,dy = 0$ for all $x \in \boldsymbol{R}$. Show that $f \equiv 0$.

See Spring 2020 Problem 1. ■

Spring 2019

Problem 1. Consider the Hilbert space $L^2(\boldsymbol{T})$ of complex valued square integrable functions with the inner product given by

$$(f,g) = \int_{\boldsymbol{T}} \overline{f(x)} g(x)\, dx$$

(a) For all $\varphi \in \boldsymbol{R}$, define $g_\varphi(\theta) = \sin(\theta+\varphi)$ for $\theta \in [0, 2\pi]$. Let V be the closed linear span of $\{\, g_\varphi \,|\, \varphi \in \boldsymbol{R} \,\}$. Show that V is two-dimensional.

(b) Find $k : \boldsymbol{T} \times \boldsymbol{T} \to \boldsymbol{C}$ such that for all $f \in L^2(\boldsymbol{T})$ the integral operator K defined by

$$Kf(x) = \int_{\boldsymbol{T}} k(x,y) f(y)\, dy$$

satisfies

$$||Kf - f|| = \inf \left\{ \, ||g - f|| \;\middle|\; g \in V \, \right\}$$

(a) We claim $\{\sin(\theta), \cos(\theta)\}$ is a basis for V, from which it follows that V is two dimensional.

Note that for any $g \in V$, we can write

$$\begin{aligned} g(\theta) &= \sum_{\varphi \in \mathcal{A}} g_\varphi(\theta) \\ &= \sin\theta \sum_{\varphi \in \mathcal{A}} \cos\varphi + \cos\theta \sum_{\varphi \in \mathcal{A}} \sin\varphi \end{aligned}$$

for some finite collection of real numbers $\mathcal{A}$ by the angle sum and difference formulas.

We show that given any $\alpha, \beta \in \boldsymbol{R}$, we may write

$$\alpha = \sum_{\varphi \in \mathcal{A}_\alpha} \cos\varphi \qquad \text{and} \qquad \beta = \sum_{\varphi \in \mathcal{A}_\beta} \sin\varphi$$

Note that it is clear that $g_0(\theta) = \sin(\theta)$, $g_\pi(\theta) = -\sin(\theta)$, $g_{\pi/2}(\theta) = \cos(\theta)$, and $g_{-\pi/2}(\theta) = -\cos(\theta)$ so we can obtain any integer α, β by repeated addition of these elements.

Let $\tilde{\alpha}$ be the fractional part of α and $\tilde{\beta}$ be the fractional part of β. If we can obtain any fractional part, then we are done and $\{\sin\theta, \cos\theta\}$ is a basis of V.

Let $0 \leqslant \tilde{\alpha} \leqslant 1$ and choose φ such that $\cos(\varphi) = \tilde{\alpha}/2$. Then

$$g_\varphi(\theta) + g_{-\varphi}(\theta) = 2\cos\varphi = \tilde{\alpha}$$

Similarly, let $0 \leqslant \tilde{\beta} \leqslant 1$ and choose φ such that $\sin(\varphi) = \tilde{\beta}/2$. Then

$$g_\varphi(\theta) + g_{\pi-\varphi}(\theta) = 2\sin\varphi = \tilde{\beta}$$

as desired. □

(b) We are looking for the unique element in V closest to range of the integral operator K, which is a subset of $L^2(\boldsymbol{T})$.

■

Problem 6. Let $\Omega = (0,1) \subset \boldsymbol{R}$. For $\overline{u} := \int_\Omega u\,dx$, show that

$$||u - \bar{u}||_{L^\infty(0,1)} \leqslant ||u'||_{L^2(0,1)} \qquad \forall\, u \in W^{1,1}(\Omega).$$

We have

$$\begin{aligned} u(x) - \overline{u} &= \int_0^1 u(x) - u(y)\,dy \\ &= \int_0^1 \int_y^x u'(\eta)\,d\eta && (W^{1,p} \text{ FTC (Brezis 8.2)}) \\ &\leqslant \int_0^1 \int_y^x |u'(\eta)|\,d\eta \leqslant \int_0^1 \int_0^1 |u'(\eta)|\,d\eta && (\text{sup } x \text{ and } y) \\ &= ||u'||_{L^1} \leqslant ||u'||_{L^2} && (\text{Hölder's with 1 and } u') \end{aligned}$$

sup over x on both sides to obtain the inequality. ■

Fall 2018

Problem 1. Let $f : [0, 1] \to \boldsymbol{R}$ be a continuous function. Prove that

$$\lim_{n\to\infty} \int_0^1 f(x)\sin(n\pi x)\,dx = 0$$

(c.f. Spring 2017 Problem 5)

This is the well-known "Riemann-Lebesgue lemma", it is more generally a statement about the coeficient decay of the L^2 Fourier expansion for f. We will prove this result in three different ways. Note that all proofs hold for $e^{in\pi x}$ in place of $\sin(n\pi x)$ which is a more general result.

The first proof is the "unsophisticated" proof, presented before the notions of measure theory, using only undergraduate notions of continuity. The other two proofs use approximation properties of continuous functions.

Proof. Let $x = \xi + \frac{1}{n}$ for an integer n, then

$$\int_0^1 f(x)\sin(n\pi x)\,dx = -\int_{-1/n}^{1-1/n} f\left(\xi + \frac{1}{n}\right)\sin(n\pi\xi)\,d\xi$$

Consequently

$$\begin{aligned}
2\int_0^1 f(x)\sin(n\pi x)\,dx &= \int_0^1 f(x)\sin(n\pi x)\,dx - \int_{-1/n}^{1-1/n} f\left(\xi + \frac{1}{n}\right)\sin(n\pi\xi)\,d\xi \\
&= \int_0^{1-1/n} (f(x) - f\left(x + \frac{1}{n}\right))\sin(n\pi x)\,dx \\
&\quad + \int_{1-1/n}^1 f(x)\sin(n\pi x)\,dx \\
&\quad - \int_{-1/n}^0 f\left(x + \frac{1}{n}\right)\sin(n\pi x)\,dx \\
&=: I + J - K
\end{aligned}$$

Since f is continuous on $[0, 1]$, it is uniformly continuous and hence bounded by a constant $M > 0$. Then

$$|I| \leqslant M\int_{1-1/n}^1 |\sin(n\pi x)|\,dx \leqslant \frac{M}{n}$$

and

$$|J| \leqslant M\int_{-1/n}^0 |\sin(n\pi x)|\,dx \leqslant \frac{M}{n}$$

Since f is uniformly continuous, given any $\varepsilon > 0$ we may choose n large enough so that $|f(x) -$

$f(x+1/n)| < \varepsilon/3$ for all $x \in [0,1]$. This shows that

$$\begin{aligned} |K| &\leqslant \int_0^{1-1/n} |f(x) - f\left(1+\frac{1}{n}\right)||\sin(n\pi x)|\,dx \\ &< \frac{\varepsilon}{3}\int_0^{1-1/n} |\sin(n\pi x)|\,dx < \frac{\varepsilon}{3} \end{aligned}$$

Choose n to be the smallest such n such that $|I|, |J|$ and $|K|$ are all less than $\varepsilon/3$. Then

$$\left|\int_0^1 f(x)\sin(n\pi x)\,dx\right| \leqslant |I| + |J| + |K| < 3\frac{\varepsilon}{3} = \varepsilon$$

as desired. ■

Proof. By the Weierstrauss approximation theorem f can be approximated by a sequence of polynomials $p_k : [0,1] \to \boldsymbol{R}$ such that $||f - p_k||_{L^\infty} \to 0$ as $k \to \infty$. Let ε be given and choose N such that $||f - p_k||_{L^\infty} < \varepsilon/2$ whenever $k \geqslant N$. Then a simple estimate shows

$$\begin{aligned} \left|\int_0^1 (p_k(x) - f(x))\sin(n\pi x)\right| &\leqslant \int_0^1 |p_k(x) - f(x)||\sin(n\pi x)|\,dx \\ &\leqslant ||p_k - f||_{L^\infty}\,||\sin(n\pi x)||_{L^1} \\ &< \varepsilon \end{aligned}$$

Next, for any fixed k we may choose an n depending on k such that

$$\frac{1}{n\pi}\left[|p_k(0) - p_k(1)| + \int_0^1 |p_k'(x)|\,dx\right] < \varepsilon/2$$

since p_k is a polynomial and hence bounded on $[0,1]$.

Note that a polynomial p_k can be differentiated and integrated on the interval $[0,1]$. Hence we estimate, integrating by parts and choosing a sufficient n depending on k as above

$$\begin{aligned} \left|\int_0^1 f(x)\sin(n\pi x)\,dx\right| &\leqslant \left|\int_0^1 (f(x) - p_k(x))\sin(n\pi x)\right| + \left|\int_0^1 p_k(x)\sin(n\pi x)\right| \\ &< \frac{\varepsilon}{2} + \left|-\frac{1}{n\pi}p_k(x)\cos(n\pi x)\Big|_0^1 + \frac{1}{n\pi}\int_0^1 p_k'(x)\cos(n\pi x)\,dx\right| \\ &\leqslant \frac{\varepsilon}{2} + \frac{1}{n\pi}\left[|p_k(0) - p_k(1)| + \int_0^1 |p_k'(x)|\,dx\right] \\ &< \varepsilon \end{aligned}$$

as desired. ■

Proof. Since f is continuous on $[0,1]$ it is $L^1([0,1])$ and can be approximated by a sequence of simple functions $\varphi_k(x) \nearrow f(x)$ as $k \to \infty$. Since simple functions are dense in L^1, given any $\varepsilon > 0$ we may choose an N large enough that $||f - \varphi_k||_{L^1} < \varepsilon/2$ whenever $k \geqslant N$ and hence by Hölder's inequality

$$\left|\int_0^1 (f(x) - \varphi_k(x))\sin(n\pi x)\,dx\right| \leqslant ||f - \varphi_k||_{L^1}\,||\sin(n\pi x)||_{L^\infty} < \varepsilon/2$$

Given any $\varepsilon > 0$ and any simple function $\varphi_k(x) := \sum_{j=1}^M \alpha_j^k \chi_{A^j}(x)$, choose n such that $\frac{2}{n\pi}\sum_j |alpha_j^k| <$

$\varepsilon/2$. Then we have the estimate

$$\begin{aligned}\left|\int_0^1 \varphi_k(x)\sin(n\pi x)\,dx\right| &= \left|\sum_{j=1}^M \alpha_j^k \int_{A^j} \sin(n\pi x)\,dx\right| \\ &\leqslant \left|\sum_{j=1}^M \alpha_j^k \int_0^1 \sin(n\pi x)\,dx\right| \\ &= \left|-\frac{1}{n\pi}\sum_{j=1}^M \alpha_j^k \cos(n\pi x)\Big|_0^1\right| \\ &\leqslant \frac{2}{n\pi}\sum_{j=1}^M |\alpha_j^k| < \frac{\varepsilon}{2}\end{aligned}$$

By the two estimates above we have that

$$\begin{aligned}\left|\int_0^1 f(x)\sin(n\pi x)\,dx\right| &\leqslant \left|\int_0^1 (f(x)-\varphi_k(x))\sin(n\pi x)\,dx\right| + \left|\int_0^1 \varphi_k(x)\sin(n\pi x)\right| \\ &< \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon\end{aligned}$$

as desired. ■

■

Problem 2. Consider the function $f : [0,1] \to \boldsymbol{R}$ defined by

$$f(x) = \begin{cases} x\log x & \text{if} x \in (0,1] \\ 0 & \text{if} x = 0 \end{cases}$$

(a) Is f Lipschitz continuous on $[0,1]$?

(b) Is f Uniformly continuous on $[0,1]$?

(c) Suppose (p_k) is a sequence of polynomial functions on $[0,1]$ converging uniformly to f. Is the set $A = \left\{ p_k \,\middle|\, k \geqslant 1 \right\} \cup \{f\}$ equicontinuous?

(a) No. $f'(x) = \log x + 1$ on $(0,1]$, and $\lim_{x\to 0^+} f'(x) = -\infty$. Suppose FSOC that f is Lipschitz with Lipschitz constant L, then $|f(x)-f(y)| \leqslant L|x-y|$ for all $x, y \in [0,1]$. Choose $x = 0$, then we have

$$\frac{|f(y)|}{y} \leqslant L$$

for all $y \in [0,1]$. Choose $y = e^{L+1}$, then

$$\frac{|e^{L+1}\log e^{L+1}|}{e^{L+1}} = L+1 > L$$

a contradiction. Therefore f is not Lipschitz.

(b) Yes. We will show that f is continuous, and therefore since f is continuous on a compact set $[0,1]$, it is uniformly continuous on $[0,1]$ (Note that this is an explicit example of uniform continuity being weaker than Lipschitz continuity).

Clearly $x\log x$ is continuous on $(0,1]$ since the product of two continuous functions is continuous. By L'Hopital's rule

$$\lim_{x\to 0^+}\frac{\log x}{1/x} = \lim_{x\to 0^+} x = 0$$

so f is continuous on $[0,1]$.

(c) Yes. Let $\varepsilon > 0$, $x\in[0,1]$ be given and choose $\delta > 0$ such that $|f(x)-f(y)| < \varepsilon/3 < \varepsilon$ whenever $|x-y| < \delta$. We will show that $|p_k(x)-p_k(y)| < \varepsilon$ whenever $|x-y|<\delta$, for all k. Choose $N\in\boldsymbol{N}$ such that

$$||p_k - f||_{L^\infty} < \frac{\varepsilon}{3} \quad\Longrightarrow\quad |p_k(x)-f(x)| < \frac{\varepsilon}{3}$$

for all $x\in[0,1]$ whenever $k\geqslant N$. We set

$$A_N := \left\{\, p_k \,\middle|\, k < N \,\right\} \subset A$$

it is clear that A_N is equicontinuous since it is a finite collection of uniformly continuous functions (take the infimum of the δ's). Because of this it is clear that if $A\setminus A_N$ is equicontinuous, then so is A. WLOG we assume N above is 1 and $A_N = \varnothing$.

We have

$$\begin{aligned}|p_k(x)-p_k(y)| &\leqslant |p_k(x)-f(x)| + |f(x)-f(y)| + |f(y)-p_k(y)| \\ &< \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3} = \varepsilon\end{aligned}$$

whenever $|x-y|<\varepsilon$ as desired. Therefore A is equicontinuous.

■

Problem 3. Show that for every $f\in C(\boldsymbol{T})$ and $\varepsilon>0$ there is an initial condition $g\in C(\boldsymbol{T})$ for which there is a solution $u(x,t)$ to the heat equation on a ring with $u(x,0)=g(x)$ and $|u(x,1)-f(x)|<\varepsilon$ for every $x\in\boldsymbol{T}$.

This is an interesting question because the heat operator is not uniquely reversible in time. This questions asks us to find a given initial data which leads to a later state prescribed by the problem, but in fact there are infinitely many such inital data satisfying the ε criterion. Any perterbation from our solution which decays sufficiently rapidly in time will also be a solution.

■

Problem 4. Consider the functions $f_N(x) = (2\pi)^{-1}\sum_{|k|\leqslant N} e^{ikx}$. Show that if $g \in L^2(\boldsymbol{T})$, then $(f_N * g) \to g$ in L^2.

A simple calculuation gives

$$\begin{aligned}(f_N * g)(x) &= \int_{\boldsymbol{T}} \frac{1}{2\pi}\sum_{|k|\leqslant N} e^{ik(x-y)} g(y)\,dy \\ &= \frac{1}{\sqrt{2\pi}}\sum_{|k|\leqslant N} e^{ikx}\frac{1}{\sqrt{2\pi}}\int_{\boldsymbol{T}} e^{-iky} g(y)\,dy \\ &= \frac{1}{\sqrt{2\pi}}\sum_{|k|\leqslant N} e^{ikx}\hat{g}_k = S_N(x)\end{aligned}$$

Since $\{e^{ikx}\}_{k\in\boldsymbol{Z}}$ is an orthonormal basis of $L^2(\boldsymbol{T})$, we can expand g as

$$g(x) = \sum_{k\in\boldsymbol{Z}} (e^{ikx}, g)_{L^2} e^{ikx}$$

where equality is taken in the L^2 sense. Because $(e^{ikx}, g)_{L^2} = \int_{\boldsymbol{T}} g(y)e^{-iky}\,dy = \hat{g}_k$ we can see that $S_N \to g$ in the L^2 sense as $N \to \infty$. ■

Problem 5. Show that for any $u \in L^1(\boldsymbol{R}^d)$

$$\lim_{h\to 0} ||u(x+h) - u(x)||_{L^1(\boldsymbol{R}^d)} = 0$$

We begin by proving a well-known lemma for definiteness.

Lemma: If $f \in L^1$ and $f_k \in L^1$ with $f_k \to f$ pointwise, then $||f - f_k||_{L^1} \to 0$ as $k \to \infty$.

Proof: Clearly $f - f_k \in L^1$. We have $|f - f_k| \leqslant |f| - |f_k|$ so by dominated convergence

$$\lim_{k\to\infty} ||f - f_k||_{L^1} = \left|\left| f - \lim_{k\to\infty} f_k \right|\right|_{L^1} = 0$$

□

We now prove the proposition. Since $C_0^\infty(\boldsymbol{R}^d)$ dense in $L^1(\boldsymbol{R}^d)$, let $\psi_k(x) \to u(x)$ as $k \to \infty$ for all $x \in \boldsymbol{R}^d$. Then by the lemma we have that $||\varphi_k - u||_{L^1} \to 0$ as $k \to 0$. Furthermore, since ψ_k is continuous $\psi_k(x+h) \to \psi_k(x)$ pointwise as $h \to 0$. Consequently $||\psi_k(x+h) - \psi_k(x)||_{L^1} \to 0$ as $h \to \infty$. Finally, note that $\psi_k(x+h) \to u(x+h)$ by a simple change of variable and the translational invarience of the Lebesgue measure on $\boldsymbol{R}^d$.

Let $\varepsilon > 0$ be given and fix some $h \in \boldsymbol{R}^d$. Choose k such that

$$||u(x+h) - \psi_k(x+h)||_{L^1} = ||\psi_k(x) - u(x)||_{L^1} < \varepsilon$$

Then we have that

$$\begin{aligned}||u(x+h)-u(x)||_{L^1} &\leqslant ||u(x+h)-\psi_k(x+h)||_{L^1} + ||\psi_k(x+h)-\psi_k(x)||_{L^1} \\ &\quad + ||\psi_k(x)-u(x)||_{L^1} \\ &< 2\varepsilon + ||\psi_k(x+h)-\psi_k(x)||_{L^1}\end{aligned}$$

Taking the limit as $h \to 0$ of the above inequality we arrive at

$$\lim_{h\to 0} ||u(x+h)-u(x)||_{L^1} < 2\varepsilon$$

by the convergence of $\psi(x+h)$ to $\psi(x)$ in L^1. Since $\varepsilon > 0$ was arbitrary the result is immediate. ■

Problem 6. Let $\Omega := \{ (x,y) \,|\, y \geqslant 0, x \in \boldsymbol{R} \}$. Let $f \in C_0^1(\boldsymbol{R}^2)$. Show that

$$\int_{\boldsymbol{R}} |f(x,0)|^2\, dx \leqslant 2 \left(\int_\Omega |f(x,y)|^2\, dxdy + \int_\Omega \left| \frac{\partial f}{\partial y}(x,y) \right|^2 dxdy \right)$$

First, let $\varphi \in C_0^1(\boldsymbol{R}^2)$ be non-negative. Then by the fundamental theorem of calculus and the vanishing of φ at infinity we find that

$$\begin{aligned}\int_{\boldsymbol{R}} \varphi^2(x,0)\, dx = -\int_{\boldsymbol{R}}\int_0^\infty \frac{\partial}{\partial y}\left[\varphi^2\right] dy\, dx &= -2\int_{\boldsymbol{R}}\int_0^\infty \varphi \frac{\partial}{\partial y}\varphi\, dy\, dx \\ &\leqslant 2\int_\Omega |\varphi| \left| \frac{\partial \varphi}{\partial y} \right| dy\, dx\end{aligned}$$

Applying young's inequality finally yields

$$\int_{\boldsymbol{R}} \varphi^2(x,0)\, dx \leqslant \int_\Omega |\varphi|^2\, dy\, dx + \int_\Omega \left| \frac{\partial \varphi}{\partial y} \right| dy\, dx \tag{$*$}$$

We now consider an arbitrary function $f \in C_0^1(\boldsymbol{R}^2)$. We have that $|f| = f_+ + f_-$ where f_+ and f_- are respectivly the positive and negative parts of f. We also have

$$|f|^2 = f_+^2 + 2f_+f_- + f_-^2 \leqslant 2(f_+^2 + f_-^2) \leqslant 2|f|^2 \tag{$**$}$$

we also note the following

$$\begin{aligned}\left|\frac{\partial f}{\partial y}\right| &= \left(\frac{\partial f}{\partial y}\right)_+ + \left(\frac{\partial f}{\partial y}\right)_- \\ &= \left(\frac{\partial}{\partial y}(f_+ - f_-)\right)_+ + \left(\frac{\partial}{\partial y}(f_+ - f_-)\right)_- \\ &= \left|\frac{\partial}{\partial y}(f_+ - f_-)\right| - \left(\frac{\partial}{\partial y}(f_+ - f_-)\right)_- + \left(\frac{\partial}{\partial y}(f_+ - f_-)\right)_- \\ &\leqslant \left|\frac{\partial f_+}{\partial y}\right| + \left|\frac{\partial f_-}{\partial y}\right|\end{aligned}$$

where we have used the fact that $|\psi| = \psi_+ + \psi_-$. This implies that

$$\left|\frac{\partial f}{\partial y}\right|^2 \leqslant 2\left(\left|\frac{\partial f_+}{\partial y}\right| + \left|\frac{\partial f_-}{\partial y}\right|\right) \leqslant 2\left|\frac{\partial f}{\partial y}\right|^2 \tag{$***$}$$

using our estimate from above.

We now estimate

$$\begin{aligned}
\int_{\boldsymbol{R}} |f(x,0)|^2\,dx &\leqslant 2\int_{\boldsymbol{R}} f_+^2(x,0) + f_-^2(x,0)\,dx \qquad \text{by } (**) \\
&\leqslant 2\int_\Omega f_+^2\,dy\,dx + 2\int_\Omega \left(\frac{\partial f_+}{\partial y}\right)^2 dy\,dx + 2\int_\Omega f_-^2\,dy\,dx \\
&\quad + 2\int_\Omega \left(\frac{\partial f_-}{\partial y}\right)^2 dy\,dx \qquad \text{by } (*) \\
&= 2\int_\Omega f_+^2 + f_-^2\,dy\,dx + 2\int_\Omega \left(\frac{\partial f_+}{\partial y}\right)^2 + \left(\frac{\partial f_-}{\partial y}\right)^2 dy\,dx \\
&\leqslant 2\int_\Omega |f|^2\,dy\,dx + 2\int_\Omega \left|\frac{\partial f}{\partial y}\right|^2 dy\,dx \qquad \text{by } (*) \text{ and } (***)
\end{aligned}$$

as desired. ■

Spring 2017

Problem 2. Suppose that X is a metric space with metric d such that every continuous functional $f : X \to \boldsymbol{R}$ is bounded. Prove that X is complete.

Suppose not, then there is a Cauchy sequence $(x_k) \in X$ such that $x_k \to x_0$ for no $x_0 \in X$ (Note that this also means NO cauchy sequence in X converges to the point $x \in \overline{X}$). Then consider the function

$$\varphi(x) := \frac{1}{d(\overline{x}, x)} \qquad \text{where} \qquad d(\overline{x}, x) := \lim_{k\to\infty} d(x_k, x)$$

Since (x_k) is Cauchy, it is bounded and by the triangle inequality together with the positive-definiteness of d, φ is well defined for all $x \in X$.

Furthermore φ is continuous since given any $x \in X$, $\varepsilon > 0$, choose $\delta < d^2(\overline{x}, x)\varepsilon/2$. Then whenever $d(x, y) < \delta$ we have

$$|\varphi(x) - \varphi(y)| = \left|\frac{d(\overline{x}, y) - d(\overline{x}, x)}{d(\overline{x}, y)d(\overline{x}, x)}\right| < \frac{d(y, x)}{d(\overline{x}, y)d(\overline{x}, x)} \leqslant \frac{d(y, x)}{\min\{d(\overline{x}, y), d(\overline{x}, x)\}^2}$$
$$< \frac{2d(x, y)}{d^2(\overline{x}, x)} < \varepsilon$$

Finally φ is unbounded since for any $M \in \boldsymbol{R}^{\geqslant 0}$ we can choose $m \in \boldsymbol{N}$ such that $\lim_{k\to\infty} d(x_k, x_m) < 1/M$ so that $\varphi(x_m) > M$. ■

Problem 3. Let A be a linear operator on a Banach space B mapping any strongly convergent sequence to a weakly convering one. Prove that A is bounded.

We assume that $A : B \to Y$ where Y is Banach (it is not stated in the problem). Since a linear operator between Banach spaces is bounded if and only if it is continuous, we show that A is continuous.

Recall that the closed graph theorem says that if $\Gamma(A) := \left\{ (x, Ax) \in B \times Y \,\middle|\, x \in B, Ax \in Y \right\}$ is closed in the product topology if and only if A is continuous.

For any sequence $x_k \to x$ in B, $Ax_k \rightharpoonup y$ in Y. $y = Ax$ since weak limits are unique and for any sequence $x_k \to x$, the sequence $(x_1, x, x_2, x, \cdots)$ also converges to x and $(Ax_1, Ax, Ax_2, Ax, \cdots)$ converges weakly to Ax since the subsequence $(Ax, Ax, Ax, \cdots)$ converges to Ax and the weak limit must agree.

Therefore (x_k, A_k) weakly converges to (x, Ax) in the product topology and $\Gamma(A)$ is weakly closed and therefore strongly closed. Therefore A is continuous and hence bounded. ■

Problem 5. Prove the Riemann-Lebesgue Lemma, namely if $f \in L^1(\boldsymbol{R}^d)$, then $\hat{f}(\xi) \to 0$ as $|\xi| \to 0$, where

$$\hat{f}(\xi) = \int_{\boldsymbol{R}^d} e^{-2\pi i\xi\cdot x} f(x)\, dx$$

(c.f. Fall 2018, Problem 1)

Since $f \in L^1(\boldsymbol{R}^d)$, it can be approximated in norm simple functions of $\psi_k \in C_0^\infty(\boldsymbol{R}^d)$ such that $\psi_k \to f$ (in L^1 as $k \to \infty$.

We estimate by Hölder's inequality

$$\left| \int_{\boldsymbol{R}^d} e^{-2\pi i \xi \cdot x}(f(x) - \psi_k(x))\, dx \right| \leqslant ||f - \psi_k||_{L^1} \left|\left|e^{-2\pi i \xi \cdot x}\right|\right|_{L^\infty} = ||f - \psi_k||_{L^1}$$

We also have

$$\int_{\boldsymbol{R}^d} \psi_k(x) e^{-2\pi i \xi \cdot x}\, dx = \frac{1}{((2\pi)^{d/2} i|\xi|)^k}\,()$$

Let $\varepsilon > 0$ be given and choose k such that $||f - \psi_k||_{L^1} < \varepsilon$. Then

$$\begin{aligned} \left| \int_{\boldsymbol{R}^d} f(x) e^{-2\pi i \xi \cdot x}\, dx \right| &\leqslant \left| \int_{\boldsymbol{R}^d} (f(x) - \psi_k(x)) e^{-2\pi i \xi \cdot x}\, dx \right| + \left| \int_{\boldsymbol{R}^d} \psi_k(x) e^{-2\pi i \xi \cdot x}\, dx \right| \\ &< ||f - \psi_k||_{L^1} + \end{aligned}$$

Taking the limit as $|\xi| \to \infty$ and noting the arbitraryness of ε, we obtain our result. ■

Problem 6. Give an example of a sequence of functions that converges weakly in L^2, strongly in L^1, but does not converge strongly in L^2. Be sure to justify your assertions.

Choose the sequence $\varphi_k(x) = k^{1/2} \mathbb{1}_{[0,1/k]}(x)$. We may take the domain to be $[0, 1]$ with the obvious extension to $\boldsymbol{R}$ or $\boldsymbol{T}$.

First, note that $\varphi_k \in L^2 \cap L^1$ since

$$\begin{aligned} ||\varphi_k||_{L^2} &= \int_0^{1/k} k\, dx = 1 < \infty \\ ||\varphi_k||_{L^1} &= \int_0^{1/k} k^{1/2}\, dx = \frac{1}{k^{1/2}} < \infty \end{aligned}$$

First, we show that $\varphi_k \rightharpoonup 0$ in L^2. Observe that for any $\varphi \in L^2$ we have

$$\begin{aligned} \int_0^{1/k} k^{1/2} \varphi(x)\, dx &\leqslant \left(\int_0^{1/k} k\, dx \right)^{1/2} \left(\int_0^{1/k} \varphi^2(x)\, dx \right)^{1/2} \\ &\leqslant \frac{1}{k^{1/2}} ||\varphi_k||_{L^2} = \frac{1}{k^{1/2}} \end{aligned}$$

where we have applied Hölder's inequality twice; the first time with exponents 2 and 2, and the second time with exponents ∞ and 1.

Taking the limit of both sides proves the assertion that $\varphi_k \rightharpoonup 0$ in L^2.

Next, take the limit of the L^1 norm of φ_k to see that $||\varphi_k||_{L^1} \to 0$ as $k \to 0$ and hence $\varphi_k \to 0$ strongly in L^1.

Finally, because the L^2 norm is constant, we cannot have strong convergence to zero in L^2. ■

Fall 2017

Problem 1. Prove that every metric subspace of a separable metric space is separable.

Let S be a seperable metric space with dense subset D. That is, for every $x \in S$ there is a sequence $(x_k)_{k=1}^{\infty}$ with $x_k \in D$ for all k with $x_k \to x$ as $k \to \infty$. Let $M \subset S$ be a metric subspace of S. Then every $x \in M$, $x \in S$ and there is a sequence as before and hence D is dense in M. ■

Problem 2. Let X be a Banach space with dual space X^* and let $A \subset X$ be a linear subspace. Define the annihilator $A^{\perp} \subset X^*$ by

$$A^{\perp} := \left\{ f \in X^* \,\middle|\, f(x) = 0 \text{ for all } x \in A \right\}$$

Prove that A is dense in X if and only if $A^{\perp} = \{0\}$.

Suppose that A is dense in X and suppose for the sake of contradiction that $\varphi \in X^*$ is a non-trivial bounded linear functional in $A^{\perp}$. Then $\varphi(x) = 0$ for all $x \in A$. Since φ is non-trivial, there is a point $y \in X$ such that $\varphi(y) \neq 0$. Since A is dense in X, there is a sequence $(y_k)_{k=1}^{\infty}$ with $y_k \in A$ for all k such that $y_k \to y$ as $k \to \infty$. Since φ is bounded it is continuous and

$$0 = \lim_{k\to\infty} \varphi(y_k) = \varphi\left(\lim_{k\to\infty} y_k\right) = \varphi(y)$$

which is a contradiction. Therefore $A^{\perp} = \{0\}$.

Conversely, suppose that $A^{\perp} = \{0\}$ and suppose for the sake of contradiction that A is not dense in X. Then $\overline{A}$ is a proper subset of X and a well known corollary to Hahn-Banach Theorem asserts that for every proper linear subspace A of X there is a non-trivial bounded linear functional which vanishes identically on A. Therefore $A^{\perp} \neq \{0\}$ which is a contradiction. ■

Spring 2016

Problem 1. Let $f(x)$ be a continuous function on $\boldsymbol{R}$ such that for any polynomial $P(x)$ we have

$$\int_{\boldsymbol{R}} f(x)P(x)\,dx = 0$$

Show that $f(x)$ is identically zero.

This question is wrong as stated, we produce a counter example. Thank you to Alvin Chen for finding a counter example.

Along the lines of why this doesn't work see F2015:P6 which adds that $f \in \mathcal{S}(\boldsymbol{R})$. Our counter example is not Schwartz, and the mechaism we abuse uses the fact that we can get oscillations of higher and higher frequencies to occur further and further away from the origin. Obviously if f is Schwartz or we are on a compact set this cannot happen.

This counter example is taken from mathcounterexamples.net.

Consider the integral

$$J_k = \int_0^\infty x^k e^{-(1-i)x}\,dx \leqslant \int_0^\infty x^k e^{-x}\,dx < \infty \tag{1}$$

repeated integration by parts gives the closed formula

$$J_k = \frac{k!}{(1-i)^{k+1}} = \frac{k! e^{i\frac{\pi}{4}(k+1)}}{2^{\frac{k+1}{2}}}$$

Then if we choose $k = 4n+3$ for $n \in \boldsymbol{Z}$, then $J_k \in \boldsymbol{R}$ and the imaginary part of the integral (1) is zero from which we obtain the formula

$$\int_0^\infty x^{4p+3} e^{-x} \sin(x)\,dx = 0$$

Therefore after the u-substitution $x = u^{1/4}$ we arrive at the integral

$$\int_0^\infty u^p \sin(u^{1/4}) e^{-u^{1/4}}\,du = 0$$

for any $p \geqslant 0$.

Now, define the function

$$f(x) = \begin{cases} \sin(x^{1/4})e^{-x^{1/4}}, & x > 0 \\ -\sin(x^{1/4})e^{-x^{1/4}}, & x \leqslant 0 \end{cases}$$

this function is seen to be continuous on $\boldsymbol{R}$. By the above calculation, for any monomial x^k, $\int_{\boldsymbol{R}} x^k f\,dx = 0$ and the result follows. ■

Problem 2. Let M be a multiplication on L^2 defined by

$$Mf(x) = m(x)f(x)$$

where $m(x)$ is continuous and bounded. Prove that M is a bounded operator on $L^2(\boldsymbol{R})$ and that its spectrum is given by

$$\sigma(M) = \overline{\{ m(x) \,|\, x \in \boldsymbol{R} \}}$$

Can M have eigenvalues?

M is a bounded operator on L^2 by Cauchy-Schwartz, in particular

$$||Mf||_{L^2} = \int_{\boldsymbol{R}} m(x)f(x)\,dx \leqslant ||m||_{L^\infty} \left(\int_{\boldsymbol{R}} |f(x)|^2\,dx \right)^{1/2} = ||m||_{L^\infty} \, ||f||_{L^2}$$

Next we compute the spectrum of M. Let $g \in L^2$, then

$$(m(x) - \lambda)f(x) = g(x)$$

if $(m(x) - \lambda) \neq 0$ for all $x \in \boldsymbol{R}$ and in this case

$$f(x) = \frac{g(x)}{m(x) - \lambda}$$

However we also require that f be in L^2, which happens when $(m(x) - \lambda)^{-1} \in L^\infty$ in which case $f \in L^2$ by Cauchy-Schwartz. This means

$$\sup_{x \in \boldsymbol{R}} \frac{1}{|m(x) - \lambda|} \leqslant C, \qquad \frac{1}{C} \leqslant \inf_{x \in \boldsymbol{R}} |m(x) - \lambda|$$

which means for f to be in L^2 we require that $x \notin \overline{\{ m(x) \,|\, x \in \boldsymbol{R} \}}$. Note that this also gives that $\ker m(x) - \lambda = \{0\}$ since $(m(x) - \lambda)f = 0 \Rightarrow f = 0$. Therefore $\sigma(M) = \overline{\{ m(x) \,|\, x \in \boldsymbol{R} \}}$.

Finally, M can have eigenvalues, for example when $m = 1$, then 1 is an eigenvalue of M. ■

Problem 3. Show that the closed unit ball of a Hilbert space H is compact if and only if $\dim H < \infty$.

Let $B := \overline{\boldsymbol{B}_1(0)} \subset H$ be compact and suppose, for the sake of contradiction, that $\dim H = \infty$. Then H has an orthonormal basis $(u_\alpha)_{\alpha \in \mathcal{I}}$ for some infinite set $\mathcal{I}$. Certainly $u_\alpha \in B$ for all $\alpha \in \mathcal{I}$. Since H is a metric space, and B is compact, B is sequentially compact and every sequence has a convergent subsequence. Let $(u_k)_{k=1}^\infty$ be such a convergent subsequence of basis elements. We compute

$$||u_m - u_n||^2 = ||u_m||^2 - (u_m, u_n) - (u_n, u_m) + ||u_n||^2 = 2$$

so $(u_k)_{k=1}^\infty$ is not Cauchy and hence not convergent. This is a contradiction since we chose $(u_k)_{k=1}^\infty$ to be convergent. Hence $\dim H < \infty$.

Conversely, suppose that $N := \dim H < \infty$. Since H is Hilbert there is an orthonormal basis $(u_k)_{k=1}^N$ of H. We show that $B := \overline{\boldsymbol{B}_1(0)} \subset H$ totally bounded, and hence compact. Completeness is a triviality, every closed subset of a Banach space is complete.

Let $\varepsilon > 0$ be given. WLOG assume that $\varepsilon < 1$ (since otherwise $B \subset \boldsymbol{B}_\varepsilon(0)$). Construct the following

ε-net. Let

$$\mathcal{N}_k := \left\{ \pm\frac{\varepsilon}{N} m u_k \,\middle|\, 0 \leqslant m \leqslant \lceil \frac{N}{\varepsilon} \rceil, m \in \boldsymbol{Z}^{\geqslant 0} \right\}$$

Each $\mathcal{N}_k$ can be thought of as a seqeunce of "points" in each "coordinate direction" using the geometric intuition about $\boldsymbol{R}^d$. Then consider

$$\mathcal{N} := \bigoplus_{k=1}^{N} \mathcal{N}_k$$

the direct sum of $\mathcal{N}_k$. Clearly $\mathcal{N}$ is finite with $|\mathcal{N}| \leqslant \lceil \frac{N}{\varepsilon} \rceil^N$, we must show that $\mathcal{N}$ is an ε-net.

Indeed, let $\varphi \in B$, then $\varphi = \sum_{k=1}^{N} (\varphi, u_k) u_k$. We define $\varphi^{\mathcal{N}} = \sum_{k=1}^{N} m_k u_k$ by choosing m_k such that $\frac{\varepsilon}{N} m \leqslant (\varphi, u_k) \leqslant \frac{\varepsilon}{N}(m+1)$ which we can do since $\varepsilon < 1$ and $(\varphi, u_k) \leqslant 1$ (by Parseval's equality). Then

$$\varphi - \varphi^{\mathcal{N}} = \sum_{k=1}^{N} ((\varphi, u_k) - m_k) u_k$$

by our choice of m_k, we can see that $(\varphi, u_k) - m_k \leqslant \frac{\varepsilon}{N}$. Therefore

$$\left|\left|\varphi - \varphi^{\mathcal{N}}\right|\right| \leqslant \sum_{k=1}^{N} |(\varphi, u_k) - m_k| \, ||u_k|| \leqslant \sum_{k=1}^{N} \frac{\varepsilon}{N} = \varepsilon$$

so $\varphi \in \boldsymbol{B}_\varepsilon(\varphi^{\mathcal{N}})$ so $\mathcal{N}$ is an ε-net of B and B is compact. ■

Problem 4. Suppose that f is a function in the Schwartz space $\mathcal{S}(\boldsymbol{R})$ which satisfies the normalizing condition $||f||_{L^2(\boldsymbol{R})} = 1$. Let $\hat{f}$ denote the Fourier transform of f. Show that

$$\frac{1}{16\pi^2} \leqslant \left|\left|\xi \hat{f}\right|\right|^2_{L^2(\boldsymbol{R})} ||xf||^2_{L^2(\boldsymbol{R})}$$

(c.f. Fall 2019 Problem 4)

Note: Babsen's notes say this problem has a bad normalization but this is incorrect. The problem as stated is correct

This is known as Heisenberg's uncertainty principle. We have

$$1 = \int_{\boldsymbol{R}} f^2 \, dx = -\int_{\boldsymbol{R}} x \frac{d}{dx} f^2 \, dx = -2 \int_{\boldsymbol{R}} x f f' \, dx$$

then

$$\begin{aligned} 1 = ||f||_{L^2(\boldsymbol{R})} &\leqslant 2 \int_{\boldsymbol{R}} |x||f||f'| \, dx \\ &\leqslant 2 \, ||xf||_{L^2(\boldsymbol{R})} \, ||f'||_{L^2(\boldsymbol{R})} \end{aligned}$$

by Cauchy Schwartz. We have by Plancheral's identity and the properties of the Fourier transform

$$||f'||_{L^2(\boldsymbol{R})} = 2\pi||\xi\hat{f}||_{L^2(\boldsymbol{R})}$$

which is the desired identity. ■

Problem 6. Let H be a Hilbert space and let U be a unitary operator, that is surjective and isometric, on H. Let $I = \{ v \in H \,|\, Uv = v \}$ be the subspace of invariant vecotrs with respect to U.

(a) Show that $\{ Uw - w \,|\, w \in H \}$ is dense in $I^\perp$ and that I is closed.

(b) Let P be the orthogonal projection onto I. Show that

$$\frac{1}{N}\sum_{k=1}^{N} U^k v \to Pv$$

The content of this question is the von Neumann ergodic theorem, which is proved in Hunter-Nachtergaele, Theorem 8.35 [], and Rudin III, Theorem 12.44 [].

(a) Let $v_k \to v$, $v_k \in I$. Since U is continuous

$$v = \lim_{k\to\infty} v_k = \lim_{k\to\infty} Uv_k = U\left(\lim_{k\to\infty} v_k\right) = Uv$$

so I is closed.

Next, let $w \in I^\perp$, $w_k \to w$ in H, and consider the sequence $m_k = (U - I)^* w_k = (U^* - I)w_k$. Clearly $m_k \in H$ and then $w_k = (U - I)m_k \in \{ Uv - v \,|\, v \in H \}$. Since $w_k \to w$ by construction the set is dense in $I^\perp$. □

(b) We begin by noting that $H = \text{ran}P \oplus \ker P$, so we may prove the theorem for the kernel and the range seperatly. The assertion is trivial on $\text{ran}P = I$ since $Uv = v$.

By part (a), let $w^\perp = (U - I)v \in \ker P = I^\perp$ for some $v \in H$. Then we have

$$\frac{1}{N}\sum_{k=1}^{N} U^k w^\perp = \frac{1}{N}\sum_{k=1}^{N}(U^{k+1} - U^k)v = \frac{1}{N}(U^{N+1} - U)v$$

which vanishes in the limit that $N \to 0$.

Again by part (a), let $(U - I)v_j = w_j \to w^\perp \in \ker P$, where $(U - I)w_j \in \ker P$ for all j, $v_j \in H$. Then we have

$$\lim_{N\to\infty}\frac{1}{N}\left|\left|\sum_{k=1}^{N} U^k w^\perp\right|\right| \leqslant \limsup_{N\to\infty}\frac{1}{N}\left(\left|\left|\sum_{k=1}^{N} U^k(w^\perp - w_j)\right|\right| + \left|\left|\sum_{k=1}^{N} U^k w_j\right|\right|\right) \leqslant ||w^\perp - w_j||$$

where the first norm can be estimated using the fact that U is unitary and hence bounded on H with operator norm $||H|| = 1$, and the second norm vanishes. Taking the limit that $j \to 0$ concludes the proof.

■

Spring 2015

Problem 1. Let $\mathcal{H}$ be a seperable Hilbert space. A sequence (x_k) in $\mathcal{H}$ converges in the Cesàro sense to $x \in \mathcal{H}$ if the averages of its partial sums converge strongly to x, i.e., if

$$\overline{x}_N = \frac{1}{N}\sum_{n=1}^{N} x_n \to x, \qquad \text{as } N \to \infty$$

(a) Prove that if (x_n) converges strongly to $x \in \mathcal{H}$, then (x_n) also converges in the Cesàro sense to x.

(b) Give an example of a sequence which converges in the Cesàro sense but does not converge weakly.

(c) Give an example of a sequence which converges weakly but does not converge in the Cesàro sense.

This question uses an Incorrect definition of Cesàro summation.
Actual Cesàro summation is defined as

$$\overline{x}_N = \sum_{k=1}^{N}\sum_{j=1}^{k} a_j$$

We solve the question using the given definition, which is convergence in the average, and not the real definition of Cesàro convergence.

(a) Without loss of generality let $x_k \to 0$ (replace x_k by $x_k - x$) in $\mathcal{H}$, $M \geqslant N$, then

$$||\overline{x}_M - \overline{x}_N|| \leqslant \frac{1}{N}\sum_{k=N}^{M} ||x_k|| \leqslant \sup_{M \leqslant k \leqslant N} ||x_k||$$

Taking the limit as $M, N \to \infty$, together with the strong convergence of $x_k \to 0$ shows the sequence is Cauchy and hence converges. □

(b) Consider the sequence $\{1, -1, 1, -1, \cdots\}$ in $\boldsymbol{R}$. Recall that in finite dimensional spaces weak and strong convergence are equivalent, so this sequence clearly doesn't converge. It is easy to compute the means and find that they are $\{1, 0, \frac{1}{3}, 0, \frac{1}{5}, \cdots\}$, and hence this sequence converges to zero in the mean sense. □

(c)

■

Problem 2. Suppose $f : [-1, 1] \to \boldsymbol{R}$ is an odd continuous function such that $f(-1) = f(1)$. Given that $\int_{-1}^{1} \sin(nx) f(x)\, dx$ for all positive integers n, show that $f \equiv 0$.

Because odd functions can be written in L^2 as the sum of sines at various frequencies, this question asks us to prove that the only odd function orthogonal to all of the basis functions for the subspace of L^2 spanned by $\{\sin(kx)\}$ is 0.

Since $f(-1) = f(1) = -f(-1)$ and f is odd, $f(1) = f(-1) = 0$. Consider the extension of f_π of f to the interval $[-\pi, \pi]$ where f_π is compactly supported on $[-1, 1]$. It is clear that f_π is continuous by the remark above together with the continuity of f. It is also clear that f_π is odd, and therefore we may expand f_π into its Fourier (sine) series (where the cosine terms drop out by orthogonality to the even basis elements of L^2)

$$f_\pi(x) = \frac{1}{\pi} \sum_{k=1}^{\infty} b_k \sin(kx)$$

where

$$b_k = \int_{-\pi}^{\pi} f_\pi(x) \sin(kx)\, dx = \int_{-1}^{1} f(x) \sin(kx)\, dx = 0$$

Since all of the Fourier components are zero, f_π is identically zero, and by the definition of f_π, $f \equiv 0$ as desired. ∎

Problem 3. Let $P(x) : \boldsymbol{R} \to \boldsymbol{R}$ be a polynomial of degree n. Show that there exists a constant C depending only on n such that $|P(\xi)| \leqslant C \int_{-1}^{1} |P(x)|^2$ for all $\xi \in (-1, 1)$.

This question is wrong as stated.

We prove that this is impossible. Suppose such a constant exists, then for $f(x) = 2^{-m} \in \Pi_0$, $m > 0$ we have $2^{-m} \leqslant C\, ||f||_{L^2}^2 = C 2^{-m+1}$ which implies that $C \geqslant 2^{m-1}$. Therefore, for any C, we can choose m large enough to violate the supposed inequality. □

However, we CAN prove what the question intended to ask, which is

$$||P(x)||_{L^\infty(-1,1)} \lesssim_n ||P||_{L^2(-1,1)}$$

for $P \in \Pi_n$, which is true.

Problem 4. Let $f_k : [0, 1] \to \boldsymbol{R}$ be a sequence of measurable functions. Suppose that

(i) $||f_k||_{L^2} \leqslant 1$ for all k.

(ii) $f_k \to 0$ a.e.

Show that

$$\lim_{k \to \infty} \int_0^1 f_k(x)\, dx = 0$$

Let $\varepsilon > 0$ be given. By Ergoroff's theorem we may choose a set $E \subset [0, 1]$ which satisfies $|E| = \varepsilon$ and $f_k \to 0$ uniformly on $[0, 1] \setminus E$.

We now estimate

$$\begin{aligned}\lim_{k\to\infty}\int_0^1 f_k(x)\,dx &= \lim_{k\to\infty}\left(\int_{[0,1]\setminus E} f_k(x)\,dx + \int_E f_k(x)\,dx\right)\\ &= \lim_{k\to\infty}\int_E f_k(x)\,dx && \text{((By unif. conv.))}\\ &\leqslant \lim_{k\to\infty}|E|^{1/2}\,||f_k||_{L^2} && \text{((By Hölder 2-2))}\\ &\leqslant \varepsilon^{1/2}\end{aligned}$$

since the L^2 norm is controlled by 1. Since ε was arbitrary this proves our result. ■

Problem 5. Find the Fourier series of the $2L$-periodic extension of

$$f(x) = \begin{cases} x & \text{if } x\in[0,L] \\ 0 & \text{if } x\in(-L,0)\end{cases}$$

Show that

$$\frac{\pi^2}{8} = \sum_{m=0}^{\infty}\frac{1}{(2m+1)^2}$$

For purely computational results like this one see Stein and Shakarchi I [], and Bleecker and Csordas []. One can easily check their answers using Desmos, Matlab, or Mathematica.

Recall that the Fourier series expansion is given by

$$f(x) = \langle f\rangle + \frac{1}{L}\sum_{k=1}^{\infty}\left[a_k\cos\left(\frac{\pi k x}{L}\right) + b_k\sin\left(\frac{\pi k x}{L}\right)\right]$$

where $\langle f\rangle := a_0/2L$ is the spatial average of f on $[-L,L]$ and

$$a_k = \int_{-L}^{L} f(x)\cos\left(\frac{\pi k x}{L}\right)\,dx, \qquad b_k = \int_{-L}^{L} f(x)\sin\left(\frac{\pi k x}{L}\right)\,dx$$

It is clear that $a_0 = \int_0^L x\,dx = L^2/2$. We integrate by parts to easily obtain

$$\begin{aligned}a_k &= \int_0^L x\cos\left(\frac{\pi k x}{L}\right)\,dx = \frac{L}{\pi k}x\sin\left(\frac{\pi k x}{L}\right)\Big|_0^L + \frac{L^2}{\pi^2k^2}\cos\left(\frac{\pi k x}{L}\right)\Big|_0^L\\ &= \frac{L^2}{\pi^2 k^2}\left((-1)^k - 1\right)\end{aligned}$$

and

$$\begin{aligned}b_k &= \int_0^L x\sin\left(\frac{\pi k x}{L}\right)\,dx = -\frac{L}{\pi k}x\cos\left(\frac{\pi k x}{L}\right)\Big|_0^L + \frac{L^2}{\pi^2k^2}\sin\left(\frac{\pi k x}{L}\right)\Big|_0^L\\ &= \frac{(-1)^{k+1}L^2}{\pi k}\end{aligned}$$

we can now construct our Fourier series, which is

$$f(x) = \frac{L}{4} + \frac{L}{\pi}\sum_{k=1}^{\infty}\left[\frac{1}{\pi k^2}\left((-1)^k - 1\right)\cos\left(\frac{\pi k x}{L}\right) + \frac{(-1)^{k+1}}{k}\sin\left(\frac{\pi k x}{L}\right)\right]$$

Choose $L = \pi$ and evaluate $f(0) = 0$, and note that in this case the sine terms vanish, the cosine terms are all 1, and we are left with

$$0 = \frac{\pi}{4} + \sum_{k=1}^{\infty}\frac{1}{\pi k^2}\left((-1)^k - 1\right) = \frac{\pi}{4} - 2\sum_{k=0}^{\infty}\frac{1}{\pi(2k+1)^2}$$

as desired. ■

Fall 2013

Problem 6. Let $\mathcal{S}(\boldsymbol{R}^d)$ be the Schwarz space. Show that $\mathcal{S}(\boldsymbol{R}^d) \subset L^p(\boldsymbol{R}^d)$ for any $1 \leqslant p \leqslant \infty$.

Recall that the Schwarz space is defined as

$$\mathcal{S}(\boldsymbol{R}^d) := \left\{ f \in C^\infty(\boldsymbol{R}^d) \,\Big|\, \sup_{x \in \boldsymbol{R}^d} |x^\alpha D^\beta f|, \quad \alpha, |\beta| = 0, 1, \cdots \right\}$$

where β is a multi-index.

We note that when $\alpha = \beta = 0$, then the Schwarz condition implies that $f \in L^\infty(\boldsymbol{R}^d)$. Furthermore, for any $\alpha = 0, 1, \cdots$, $x^\alpha f \in L^\infty(\boldsymbol{R}^d)$.

We estimate the L^p norm of f for $1 \leqslant p < \infty$. Fix any $\varepsilon > 0$, then we compute

$$\begin{aligned} \int_{\boldsymbol{R}^d} |f|^p \, dx &= \int_{\boldsymbol{B}_\varepsilon(0)} |f|^p \, dx + \int_{\boldsymbol{R}^d \setminus \boldsymbol{B}_\varepsilon(0)} |f|^p \, dx \\ &\lesssim_{\varepsilon, d} ||f||^p_{L^\infty(\boldsymbol{R}^d)} + \int_{\boldsymbol{R}^d \setminus \boldsymbol{B}_\varepsilon(0)} \frac{|x|^{dp}}{|x|^{dp}} |f|^p \, dx \\ &\lesssim_{\varepsilon, d} ||f||^p_{L^\infty(\boldsymbol{R}^d)} + \left|\left| x^d f \right|\right|^p_{L^\infty(\boldsymbol{R}^d)} < \infty \end{aligned}$$

since $1/|x|^{dp}$ is integrable away from the origin and by the remark above. Therefore $f \in L^p$ for any $1 \leqslant p \leqslant \infty$. ■

Spring 2012

Problem 1. For $u \in L^1(0,\infty)$, consider the integral

$$v(x) = \int_0^\infty \frac{u(y)}{x+y}\,dy$$

defined for $x > 0$. Show that $v(x)$ is infinitely differentiable away from the origin. Prove that $v' \in L^1(\varepsilon,\infty)$ for any $\varepsilon > 0$. Explain what happens in the limit as $\varepsilon \to 0$.

We have

$$\frac{d^k}{dx^k}v(x) = \int_0^\infty \frac{\partial^k}{\partial x^k}\frac{u(y)}{x+y}\,dy = (-1)^k k! \int_0^\infty \frac{u(y)}{(x+y)^{k+1}}\,dy$$

where the interchange is justified by dominated convergence since $u(y) \in L^1$ and $|u(y)| \geqslant |u(y)/(x+y)^k|$ for all $x > 0, y \leqslant 0, k$ (see Libeniz rule).

Since v is differentiable, v' is continuous and by the fundamental theorem of calculus

$$\int_\varepsilon^\infty v'(x)\,dx = -v(\varepsilon) = -\int_0^\infty \frac{u(y)}{\varepsilon+y}\,dy$$

We have the estimate

$$\begin{aligned}
\int_\varepsilon^\infty |v'(x)|\,dx &= \int_\varepsilon^\infty \left|\frac{d}{dx}\int_0^\infty \frac{u(y)}{x+y}\,dy\right| dx \\
&\leqslant \int_\varepsilon^\infty \int_0^\infty \frac{|u(y)|}{|x+y|^2}\,dy\,dx \\
&\leqslant \int_\varepsilon^\infty \sup_{y\in(0,\infty)} \frac{1}{|x+y|^2} \int_0^\infty |u(y)|\,dy\,dx \\
&= ||u||_{L^1(0,\infty)} \int_\varepsilon^\infty \frac{1}{x^2}\,dx \\
&= ||u||_{L^1(0,\infty)} \frac{1}{\varepsilon}
\end{aligned}$$

so that $v' \in L^1(\varepsilon,\infty)$.

In general it is not the case that $v' \in L^1(0,\infty)$; for example $u = x^{-1/2} \in L^1(0,\infty)$ but v' is non-integrable near 0. Criteria for integrability on v' amount to showing that

$$\lim_{\varepsilon\to 0}\int_0^\infty \frac{u(y)}{\varepsilon+y}\,dy = \int_0^\infty \frac{u(y)}{y}\,dy$$

For example, if $u/y \in L^1(0,\infty)$ than by monotone convergence the interchange above is justified and $v' \in L^1(0,\infty)$. ■

References