UC Davis
Applied Math
Prelim Solutions
"Back of the Napkin" style

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If you find a mistake,
please send the correct
solution to the author.

S'015 #1 (Set 1)

$$\dot{x} = -y - x(x^2+y^2)$$
$$\dot{y} = x - y(x^2+y^2)$$

Fixed pts: $0 = x - y(x^2+y^2)$

$x = y(x^2+y^2)$

$0 = -y - x(x^2+y^2)$

$y = -x(x^2+y^2)$

$y^2 = -xy(x^2+y^2)$

$y^2 = -x^2$

$x = y = 0$

$(x^*, y^*) = (0,0)$

Linear Stability:

$$J(x,y) = \begin{bmatrix} -3x^2-y^2 & -1-2xy \\ 1-2xy & -x^2-3y^2 \end{bmatrix}$$

$$J(0,0) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$\lambda = \pm i$

predicted center

$$2x\dot{x} = -2xy - 2x^2(x^2+y^2)$$
$$2y\dot{y} = 2xy - 2y^2(x^2+y^2)$$

$$2x\dot{x} + 2y\dot{y} = -2(x^2+y^2)^2$$

$$\frac{d}{dt}(x^2+y^2) = -2(x^2+y^2)^2 < 0 \quad \text{away from } (0,0)$$

$\therefore$ this is a stable spiral

S'015 #1 (set 2)

$$\dot{x} = -y + xy^2$$
$$\dot{y} = x - x^2 y$$

Fixed pts: $0 = -y + xy^2$

$0 = y(1 - xy)$

$y = 0$ or $xy = 1$

$0 = x - x^2 y$

$\underline{y=0:}$ $0 = x$

$\underline{xy=1:}$ $0 = x - x$, $0 = 0$

fixed pts: $(x^*, y^*) = (0,0)$

and $(x^*, y^*) = (t, \frac{1}{t})$, $t \neq 0$

Linear Stability:

$$J(x,y) = \begin{bmatrix} y^2 & -1 + 2xy \\ 1 - 2xy & -x^2 \end{bmatrix}$$

$$J(0,0) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\lambda = \pm i$$

predicted center

$$2x\dot{x} = -2xy + 2x^2y^2$$
$$2y\dot{y} = 2xy - 2x^2y^2$$

$$2x\dot{x} + 2y\dot{y} = 0$$

$$\frac{d}{dt}(x^2 + y^2) = 0$$

$\boxed{x^2 + y^2 = C \quad \text{closed orbits}}$

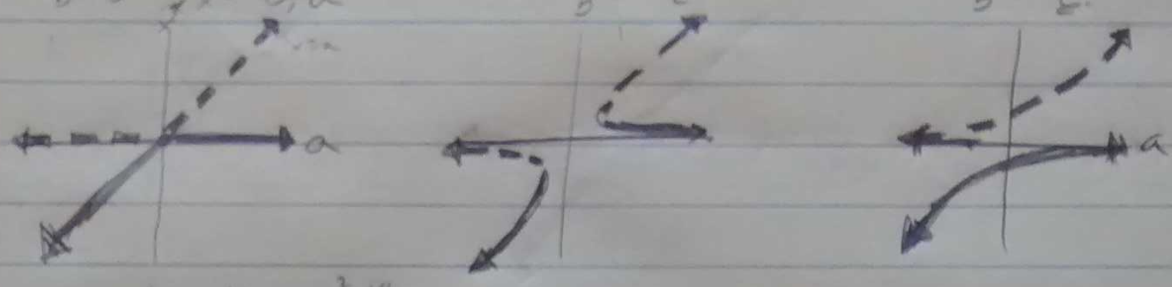
this is a center

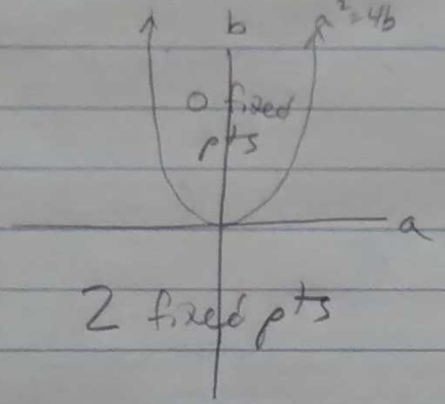
S'015 #2

$0 = x(x-a)+b = x^2 - ax + b \qquad x = \frac{a \pm \sqrt{a^2-4b}}{2}$

$\frac{\partial \dot{x}}{\partial x} = 2x - a$

$b=0 \quad x = 0, a \qquad\qquad b = +\varepsilon \qquad\qquad b = -\varepsilon$





S'015 #3 (Page 1)

$u_{tt} = u_{xx} - \mu u_t \qquad u_x(0,t) = 0 \quad u_x(\pi,t) + u(\pi,t) = 0$

$u(x,0) = \varphi(x) \quad u_t(x,0) = \psi(x) \qquad \mu > 0$

Let $u(x,t) = X(x)T(t)$

$XT'' = X''T - \mu XT' \quad , \quad X'(0) = X'(\pi) + X(\pi) = 0$

$$\frac{T''}{T} = \frac{X''}{X} - \mu\frac{T'}{T}$$

$$\frac{T''}{T} + \mu\frac{T'}{T} = \frac{X''}{X} := -n^2, \quad n \geq 0$$

$$\frac{X''}{X} = -n^2 \Rightarrow X = c_1 \cos nx + c_2 \sin nx$$

$X' = -nc_1 \sin nx + nc_2 \cos nx$

$0 = X'(0) = nc_2 \Rightarrow c_2 = 0$

$0 = X'(\pi) + X(\pi) = -nc_1 \sin \pi + c_1 \cos n\pi$

$= c_1 \cos n\pi \Rightarrow \cancel{c_1 = 0}$ or $n \in 2\mathbb{N} - 1$

$X'(x) = c_1 \cos nx \quad , n \in 2\mathbb{N} - 1.$ wlog $c_1 := 1$.

$$\frac{T''}{T} + \mu\frac{T'}{T} = -n^2 \quad , n \in 2\mathbb{N} - 1$$

$T'' + \mu T' + n^2 T = 0$

$T := e^{r_n t} \qquad r_n^2 + \mu r_n + n^2 = 0$

$$r_n = \frac{-\mu \pm \sqrt{\mu^2 - 4n^2}}{2} \qquad \therefore T(t) = a_n e^{r_{1n} t} + b_n e^{r_{2n} t}$$

$$u(x,t) = X(x)T(t) = \sum_{n \in 2\mathbb{N}-1} \left(a_n e^{r_{1n} t} + b_n e^{r_{2n} t}\right) \cos n\pi$$

S'015 #3 (Page 2)

$$\varphi(x) = u(x,0) = \sum_{n \in 2\mathbb{N}-1} (a_n + b_n)\cos n\pi$$

$$\psi(x) = u_t(x,0) = \sum_{n \in 2\mathbb{N}-1} (r_{1,n} a_n + r_{2,n} b_n)\cos n\pi$$

Project: $\varphi(x) = \sum_{n \in 2\mathbb{N}-1} \varphi_n \cos n\pi$, $\psi(x) = \sum_{n \in 2\mathbb{N}-1} \psi_n \cos n\pi$

$\varphi_n = a_n + b_n$

$\therefore \; b_n = \varphi_n - a_n$

$\psi_n = r_{1,n} a_n + r_{2,n}(\varphi_n - a_n)$

$\psi_n = (r_{1,n} - r_{2,n}) a_n + r_{2,n}\varphi_n$

$a_n(r_{1,n} - r_{2,n}) = \psi_n - r_{2,n}\varphi_n$

$$a_n = \frac{\psi_n - r_{2,n}\varphi_n}{r_{1,n} - r_{2,n}}$$

Substitute a_n, b_n into $u(x,t)$ and $k = 2n-1 \in \mathbb{N}$ to find explicit form of solution

S'015 #4

$$\int_a^x\int_a^s f(t)\,dt\,ds = \int_a^x\int_t^x f(t)\,ds\,dt$$

$$= \int_a^x f(t)\left(\int_t^x ds\right)dt$$

$$= \int_a^x (x-t)f(t)\,dt$$

$$y''+\alpha y'+c^2y=0$$

If $\alpha>0$, positive damping $\Rightarrow y\to 0$

If $\alpha<0$, negative damping

$|\alpha|>2c \Rightarrow y\to+\infty$

$0<|\alpha|<2c \Rightarrow$ oscillations of increasing amplitude

$$y''+\alpha y'+c^2y=0$$

$y(0)=0 \quad y'(0)=1$

$$y''(x)+\alpha y'(x)+c^2y(x)=0$$

$$y'(x)+\alpha y(x)+\int_0^x c^2y(t)\,dt = A$$

IC: $1+\alpha(0)+\int_0^0 c^2y(t)\,dt = A \Rightarrow A=1.$

$$y'(x)+\alpha y(x)+\int_0^x c^2y(t)\,dt = 1$$

$$y(x)+\int_0^x \alpha y(t)\,dt+\int_0^x\int_0^s c^2y(t)\,dt\,ds = x+B$$

IC: $0+\int_0^0 \alpha y(t)\,dt+\int_0^0\int_0^s c^2y(t)\,dt\,ds = 0+B \Rightarrow B=0.$

$$\therefore\ y(x)+\int_0^x \alpha y(t)\,dt+\int_0^x\int_0^s c^2y(t)\,dt\,ds = x$$

$$y(x)+\int_0^x \alpha y(t)\,dt+\int_0^x (x-t)c^2y(t)\,dt = x$$

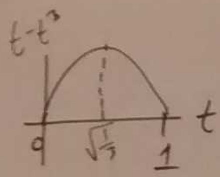
$$y(x) = x+\int_0^x[-\alpha-c^2(x-t)]y(t)\,dt$$

$$\boxed{h(x)=x \quad K(x,t)=-\alpha-c^2(x-t)}$$

Si

L

S'015 #5



$$\int_0^1 e^{x[t-t^3]}\,dt$$

$x \to \infty$:

max contribution @ $t=\sqrt{\tfrac{1}{3}}$. Expand @ $t=\sqrt{\tfrac{1}{3}}$

$$\sim \int_0^1 e^{x\left[\frac{2\sqrt{3}}{9}-\sqrt{3}\left(t-\sqrt{\frac{1}{3}}\right)^2-\left(t-\sqrt{\frac{1}{3}}\right)^3\right]}\,dt$$

$$= e^{\frac{2\sqrt{3}}{9}x}\int_0^1 e^{-\sqrt{3}x\left(t-\sqrt{\frac{1}{3}}\right)^2-x\left(t-\sqrt{\frac{1}{3}}\right)^3}\,dt$$

$$= e^{\frac{2\sqrt{3}}{9}x}\sqrt{\frac{\pi\sqrt{3}}{3x}} \quad \text{(to leading order)}$$

still needed: second term

$x \to -\infty$:

max contribution at $t=0,1$. Expand @ $t=1$:

$$=\int_0^{\frac{1}{2}} e^{x[t-t^3]}\,dt + \int_{\frac{1}{2}}^1 e^{x[-2(t-1)-3(t-1)^2]}\,dt$$

to leading order

$$\approx \int_0^{\frac{1}{2}} e^{xt}\,dt + \int_{\frac{1}{2}}^1 e^{-2x(t-1)}\,dt$$

$$= -\frac{e^{xt}}{x}\bigg|_{t=0} + \frac{e^{-2x(t-1)}}{-2x}\bigg|_{t=1}$$

$$= -\frac{1}{x}-\frac{1}{2x}$$

$$= -\frac{3}{2x}$$

still needed: second term.

$$t \to t,\ \tau$$

S'015 #6 (Page 1)

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0$$

$\theta(0) = \varepsilon$

Let $t = t + \varepsilon^2 \tau$

$$\left(\frac{\partial}{\partial t} + \varepsilon^2 \frac{\partial}{\partial \tau}\right)\theta(0) = 0$$

$$\left(\frac{\partial}{\partial t} + \varepsilon^2 \frac{\partial}{\partial \tau}\right)^2 \theta + \frac{g}{L}\left(\theta - \frac{\theta^3}{6} + \cdots\right) = 0$$

$$\theta := \varepsilon \sum_{n=0}^{\infty} \varepsilon^n \theta_n$$

$$\left(\frac{\partial^2}{\partial t^2} + 2\varepsilon^2 \frac{\partial^2}{\partial t \partial \tau} + \varepsilon^4 \frac{\partial^2}{\partial \tau^2}\right)(\varepsilon\theta_0 + \varepsilon^2\theta_1 + \varepsilon^3\theta_2 + \cdots)$$

$$+ \frac{g}{L}(\varepsilon\theta_0 + \varepsilon^2\theta_1 + \varepsilon^3\theta_2 + \cdots) - \frac{g}{6L}(\varepsilon^3\theta_0^3 + \cdots) = 0$$

$$\varepsilon\theta_0(0) + \varepsilon^2\theta_1(0) + \varepsilon^3\theta_2(0) + \cdots = \varepsilon$$

$$\left(\frac{\partial}{\partial t} + \varepsilon^2 \frac{\partial}{\partial \tau}\right)(\varepsilon\theta_0 + \varepsilon^2\theta_1 + \varepsilon^3\theta_2 + \cdots)(0) = 0$$

$$O(\varepsilon): \frac{\partial^2}{\partial t^2}\theta_0 + \frac{g}{L}\theta_0 = 0, \quad \theta_0(0) = 1, \quad \frac{\partial \theta_0}{\partial t}(0) = 0$$

$$\Rightarrow \theta_0 = A_0(\tau) e^{i\sqrt{\frac{L}{g}}t} + \bar{A}_0(\tau) e^{-i\sqrt{\frac{L}{g}}t}$$

$$A_0(0) + \bar{A}_0(0) = 1 \qquad A_0(0) - \bar{A}_0(0) = 0$$

$$\Rightarrow A_0(0) = \bar{A}_0(0) = \frac{1}{2}$$

$$O(\varepsilon^2): \frac{\partial^2}{\partial t^2}\theta_1 + \frac{g}{L}\theta_1 = 0, \quad \theta_1(0) = 0 \quad \frac{\partial \theta_1}{\partial t}(0) = 0$$

$$\Rightarrow \theta_1 = A_1(\tau) e^{i\sqrt{\frac{L}{g}}t} + \bar{A}_1(\tau) e^{-i\sqrt{\frac{L}{g}}t}$$

$$A_1(0) + \bar{A}_1(0) = 0 \qquad A_1(0) - \bar{A}_1(0) = 0$$

$$\Rightarrow A_1(0) = \bar{A}_1(0) = 0$$

$$O(\varepsilon^3): \frac{\partial^2}{\partial t^2}\theta_2 + 2\frac{\partial^2}{\partial t \partial \tau}\theta_0 + \frac{g}{L}\theta_2 - \frac{g}{6L}\theta_0^3 = 0, \quad \theta_2(0) = 0 \quad \frac{\partial \theta_2}{\partial t}(0) + \frac{\partial \theta_0}{\partial \tau}(0) = 0$$

$$\frac{\partial^2}{\partial t^2}\theta_2 + \frac{g}{L}\theta_2 = \frac{g}{6L}\theta_0^3 - 2\frac{\partial^2}{\partial t \partial \tau}\theta_0$$

$$\frac{\partial^2}{\partial t^2}\theta_2 + \frac{g}{L}\theta_2 = \frac{g}{6L}\left(A_0 e^{i\sqrt{\frac{L}{g}}t} + \bar{A}_0 e^{-i\sqrt{\frac{L}{g}}t}\right)^3 - 2i\sqrt{\frac{L}{g}}\left(A_0' e^{i\sqrt{\frac{L}{g}}t} - \bar{A}_0' e^{-i\sqrt{\frac{L}{g}}t}\right)$$

$$= \frac{g}{6L}A_0^3 e^{3i\sqrt{\frac{L}{g}}t} + \frac{g}{6L}\bar{A}_0^3 e^{-3i\sqrt{\frac{L}{g}}t} + \left(\frac{g}{2L}A_0^2\bar{A}_0 - 2i\sqrt{\frac{L}{g}}A_0'\right)e^{i\sqrt{\frac{L}{g}}t} + \left(-\frac{g}{2L}A_0\bar{A}_0^2 + 2i\sqrt{\frac{L}{g}}\bar{A}_0'\right)e^{-i\sqrt{\frac{L}{g}}t}$$

this is really grueling to do this way by hand!

need $\frac{g}{2L}A_0^2\bar{A}_0 - 2i\sqrt{\frac{L}{g}}A_0' = 0$ to avoid secular terms

I'm going to do this the easier way now.

S'015 #6 (Part 2)

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0$$

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\left(\theta - \frac{\theta^3}{6} + \cdots\right) = 0$$

$\tau := \sqrt{\frac{g}{L}}\,wt \qquad \varphi(\tau) := \theta(t)$

$$w^2\varphi'' + \left(\varphi - \frac{\varphi^3}{6} + \cdots\right) = 0$$

$w := 1 + \varepsilon^2 w_1 + \cdots$

$\varphi := \varepsilon\varphi_0 + \varepsilon^2\varphi_1 + \cdots$

$\varphi_0(0) = 1$, else $\varphi_n(0) = 0$

$\dot{\varphi}_n(0) = 0$

$$(1 + \varepsilon^2 w_1 + \cdots)^2(\varepsilon\varphi_0'' + \varepsilon^2\varphi_1'' + \varepsilon^3\varphi_2'' + \cdots)$$
$$+ \ (\varepsilon\varphi_0 + \varepsilon^2\varphi_1 + \varepsilon^3\varphi_2 + \cdots) - \frac{1}{6}(\varepsilon^3\varphi_0^3 + \cdots) = 0$$

$O(\varepsilon)$: $\varphi_0'' + \varphi_0 = 0 \qquad \varphi_0 := \frac{1}{2}(e^{it} + e^{-it})$

$O(\varepsilon^2)$: $\varphi_1'' + \varphi_1 = 0$

$O(\varepsilon^3)$: $\varphi_2'' + 2w_1\varphi_0'' + \varphi_2 - \frac{1}{6}\varphi_0^3 = 0$

$$\varphi_2'' + \varphi_2 = \frac{1}{6}\varphi_0^3 - 2w_1\varphi_0''$$
$$= \frac{1}{6}\left[\frac{1}{2}e^{it} + \frac{1}{2}e^{-it}\right]^3 - 2w_1\left[-\frac{1}{2}e^{it} - \frac{1}{2}e^{-it}\right]$$
$$= \frac{1}{48}e^{3it} + \frac{1}{16}e^{it} + \frac{1}{16}e^{-it} + \frac{1}{48}e^{-3it}$$
$$+ \ w_1 e^{it} + w_1 e^{-it}$$
$$= \frac{1}{48}\left[e^{3it} + e^{-3it}\right] + \left[\frac{1}{16} + w_1\right]e^{it} + \left[\frac{1}{16} + w_1\right]e^{-it}$$

need $\frac{1}{16} + w_1 = 0$ to avoid secular terms

$$w_1 = -\frac{1}{16}$$

$$\therefore\ w = 1 - \frac{1}{16}\varepsilon^2 + O(\varepsilon^3)$$

recall $\tau = \sqrt{\frac{g}{L}}\,wt$

so the true angular frequency is

$$\sqrt{\frac{g}{L}}\left(1 - \frac{1}{16}\varepsilon^2\right) + O(\varepsilon^3)$$