UC Davis

Applied Math

Prelim Solutions

"Back of the Napkin" style

Compiled 9/13/2016

by David Haley

If you find a mistake,
please send the correct
solution to the author.

S'014 #1 (Part 1)

$$\dot{x} = x(a-x-y)$$
$$\dot{y} = (y-2a)(x-y)$$

fixed pts:

$$0 = x(a-x-y)$$

$x=0$ or $x+y=a$

$$0 = (y-2a)(x-y)$$

$x=y$ or $y=2a$

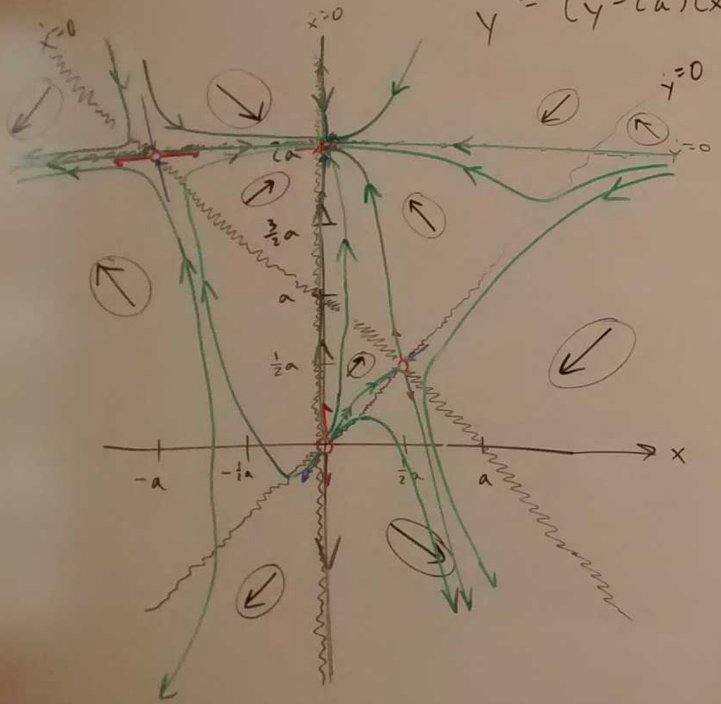
	$x=0$	$x+y=a$
$x=y$	$(0,0)$	$(\frac{1}{2}a,\frac{1}{2}a)$
$y=2a$	$(0,2a)$	$(-a,2a)$

fixed pts

S'014 #1 (Part 2)

$$\dot{x} = x(a-x-y)$$

$$\dot{y} = (y-2a)(x-y)$$



linear stability:

$$J(x,y) = \begin{bmatrix} a-2x-y & -x \\ y-2a & x-2y+2a \end{bmatrix}$$

$$J(0,0) = \begin{bmatrix} a & 0 \\ -2a & +2a \end{bmatrix}$$

$\lambda = a, 2a$ (stable for $a<0$, unstable for $a>0$)

$$\vec{u}_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$J\left(\tfrac{1}{2}a, \tfrac{1}{2}a\right) = \begin{bmatrix} -\frac{1}{2}a & -\frac{1}{2}a \\ -\frac{3}{2}a & \frac{3}{2}a \end{bmatrix}$$

$\lambda = \frac{a}{2}(1 \pm \sqrt{7})$ (saddle for $a \neq 0$)

$$\vec{u}_1 = \frac{1}{3}\begin{bmatrix} 2-\sqrt{7} \\ 3 \end{bmatrix} \qquad \vec{u}_2 = \frac{1}{3}\begin{bmatrix} 2+\sqrt{7} \\ 3 \end{bmatrix}$$

$$\approx k_1\begin{bmatrix} -1 \\ 6 \end{bmatrix} \qquad \approx k_2\begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$J(0,2a) = \begin{bmatrix} -a & 0 \\ 0 & -2a \end{bmatrix}$$

$\lambda = -a, -2a$ (stable for $a>0$, unstable for $a<0$)

$$\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \vec{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

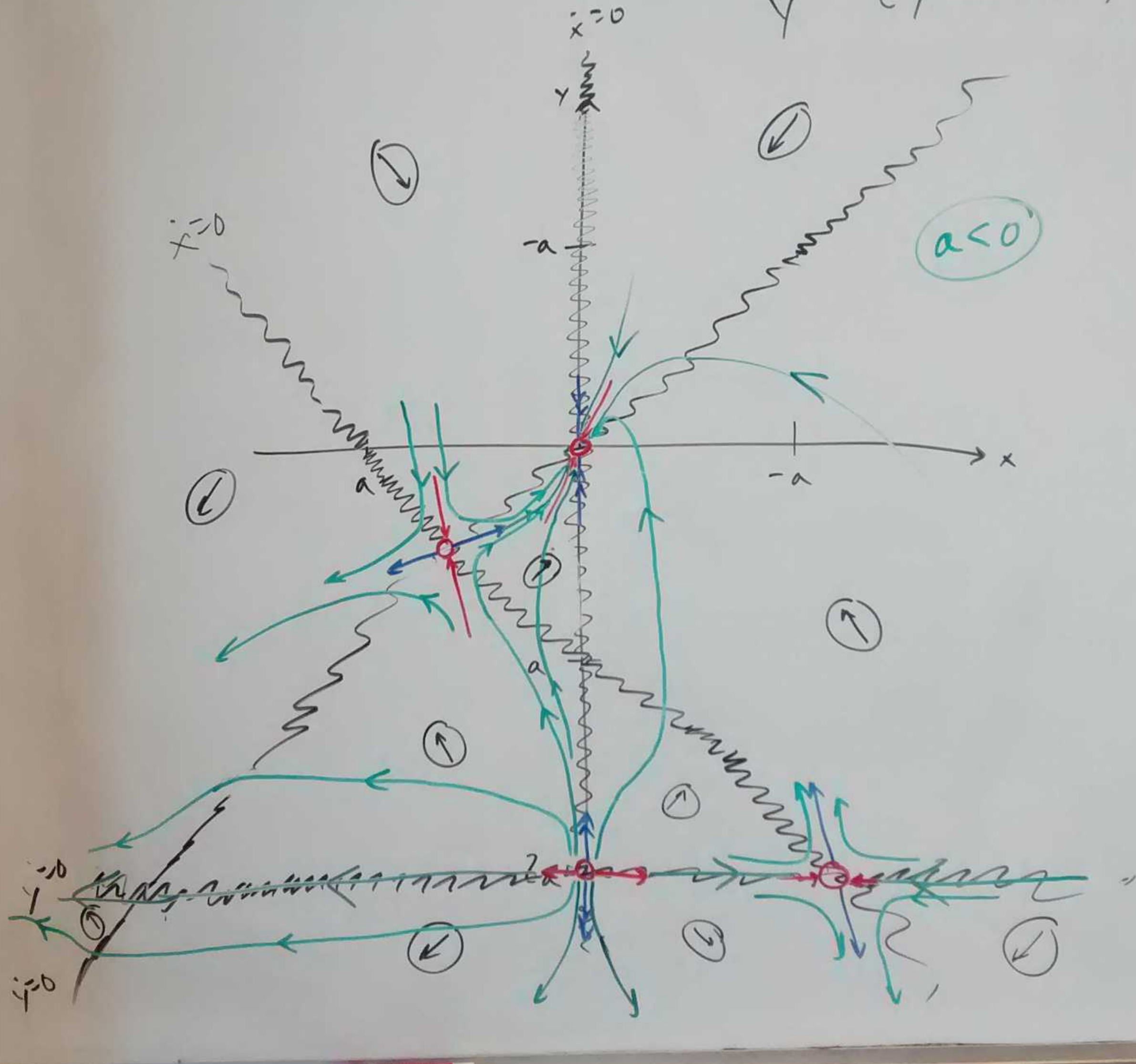
$$J(-a,2a) = \begin{bmatrix} a & a \\ 0 & -3a \end{bmatrix}$$

$\lambda = a, -3a$ (saddle for $a \neq 0$)

$$\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \vec{u}_2 = \frac{1}{\sqrt{17}}\begin{bmatrix} 1 \\ -4 \end{bmatrix}$$

S'014 #1 (Part 3)

$$\dot{x} = x(a-x-y)$$
$$\dot{y} = (y-2a)(x-y)$$



S'014 #1 (Part 4)

$$\dot{x} = x(a-x-y)$$
$$\dot{y} = (y-2a)(x-y)$$

a=0 case:

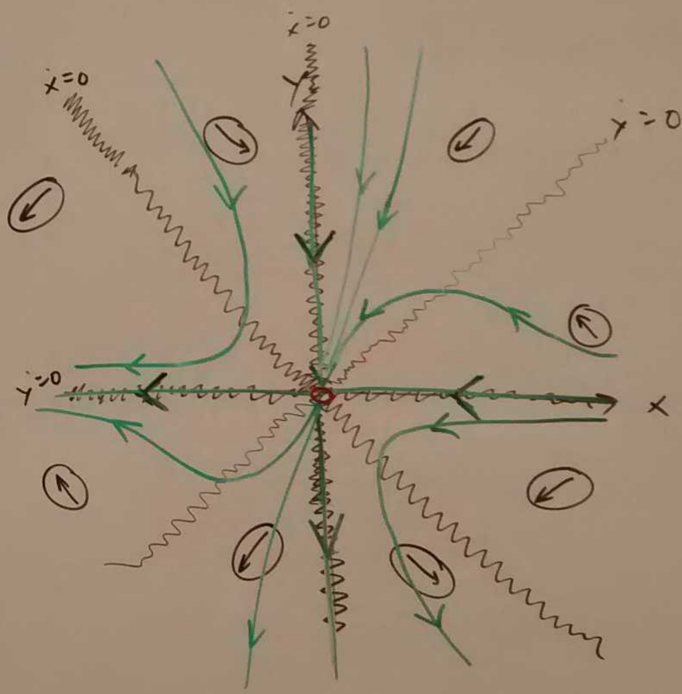
$$\dot{x} = -x^2 - xy$$
$$\dot{y} = xy - y^2$$

The previous work shows that (0,0) is the only fixed pt.

linear stability:

$$J(x,y) = \begin{bmatrix} -2x-y & -x \\ y & x-2y \end{bmatrix}$$

$$J(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ inconclusive.}$$

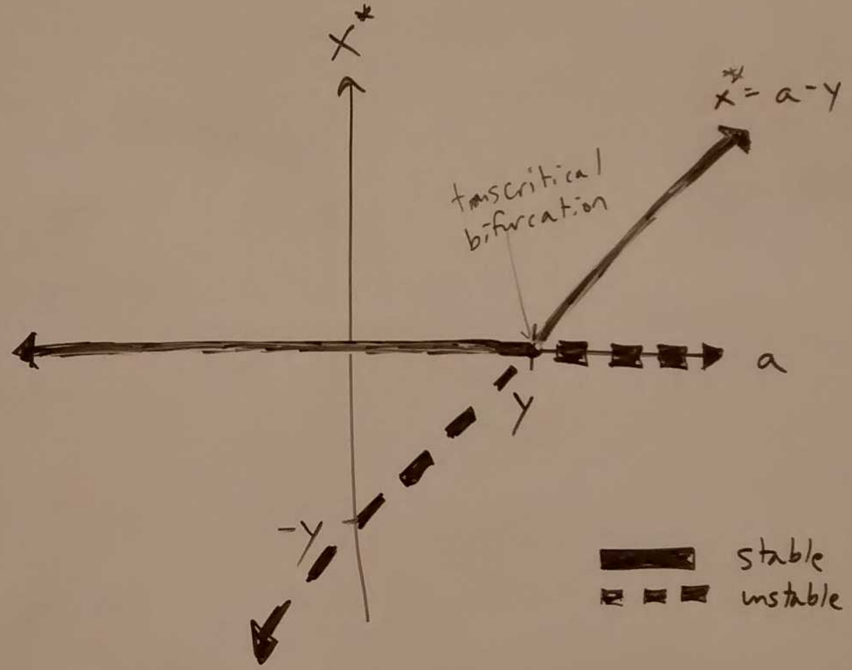


S'014 #1 (Part 5)

$$\dot{x} = x(a - x - y)$$

$$\frac{\partial \dot{x}}{\partial x} = a - 2x - y$$

fixed pt	$\frac{\partial \dot{x}}{\partial x}$
$x^* = 0$	$a - y$
$x^* = a - y$	$y - a = -x^*$



S'014 #2

Shortest Path b/w (a,b) and (c,d). I assume $a \neq c$.

that is, min $J(y) = \int_a^c \sqrt{1+(y')^2}\,dx \qquad y(a)=b,\ y(c)=d.$

E-L: $\dfrac{\partial f}{\partial y} - \dfrac{d}{dx}\dfrac{\partial f}{\partial y'} = 0$

$$0 - \frac{d}{dx}\frac{y'}{\sqrt{1+(y')^2}} = 0$$

$$\frac{y'}{\sqrt{1+(y')^2}} = C$$

$$(y')^2 = C^2(1+(y')^2)$$

$$(1-C^2)(y')^2 = C^2$$

$$(y')^2 = \frac{C^2}{1-C^2}$$

$$y' = \pm\frac{C}{\sqrt{1-C^2}} := A$$

$$\therefore\ y = Ax + B$$

That is, y is a linear function of x.

Then can just use $y = m(x-x_0)+y_0$

$$\boxed{y = \frac{d-b}{c-a}(x-a)+b}$$

S'014 #3 (Page 1)

$(xf')' + \lambda x^{-1} f = 0 \qquad f(1) = f(e) = 0$

$xf'' + f' + \lambda x^{-1} f = 0$

$f := x^n$

$n(n-1) + n + \lambda = 0$

$n^2 + \lambda = 0$ require $\lambda > 0$ to satisfy BCs

wlog let $\lambda = \pi^2 k^2$, $k > 0$

$n = \pm i\pi k$

$f(x) = c_1 \cos(\pi k \ln x) + c_2 \sin(\pi k \ln x)$

To satisfy $f(1) = 0$, $c_1 = 0$

$f(x) = c_2 \sin(\pi k \ln x)$

to satisfy $f(e) = 0$, $k \in \mathbb{N}$

Normalize: $1 = \int_1^e c_2^2 \sin^2 \pi k \ln x \, dx$

$$c_2 = \sqrt{\frac{1}{\int_1^e \sin^2 \pi k \ln x \, dx}}$$

Was this problem intended to arrive at $\sin^2 \pi k t$?

Note $\int_1^e \sin^2 \pi k \ln x \, dx \neq \int_0^1 \sin^2 \pi k t \, dt = \frac{1}{2}$

$\sin^2 \pi k t$ are the eigenfunctions of the <u>modified</u> RSL, not the original.

To expand $g(x) = 1$, Let $\phi_k(x) = c_2 \sin \pi k \ln x$, with c_2 as above,

$$g(x) = \sum_{k=1}^{\infty} \phi_k \langle \phi_k, 1 \rangle = \sum_{k=1}^{\infty} \left[c_2 \sin \pi k \ln x \int_1^e c_2 \sin \pi k \ln t \, dt \right]$$

$$= \sum_{k=1}^{\infty} \frac{\sin \pi k \ln x \int_1^e \sin \pi k \ln t \, dt}{\int_1^e \sin^2 \pi k \ln t \, dt}$$

S'014 # 4

$$\varepsilon y'' - (1+x)^2 y' + y = 0$$

$$y(0)=1, \quad y(1)=0$$

outer solution:

$O(1)$: $-(1+x^2)y' + y = 0$

$$\frac{y'}{y} = \frac{1}{1+x^2}$$

$$\ln y = \tan^{-1} x + c$$

$$y = ce^{\tan^{-1} x}$$

$$1 = y(0) = c$$

$$0 = y(1) = ce^{\frac{\pi}{4}} \Rightarrow c = 0 \text{ trivial}$$

use $y(x) = e^{\tan^{-1} x}$

and use a layer at $x=1$

inner solution:

$$x := 1 - \varepsilon^{\alpha} X \qquad Y(X) := y(x)$$

$$\varepsilon^{1-2\alpha} Y'' + \varepsilon^{-\alpha}(2-\varepsilon^{\alpha}X)^2 Y' + Y = 0$$

$$1-2\alpha = -\alpha \Rightarrow \alpha = 1$$

$$\varepsilon^{-1} Y'' + \varepsilon^{-1}(2-\varepsilon X)^2 Y' + Y = 0$$

$$Y'' + (2-\varepsilon X)^2 Y' + \varepsilon Y = 0$$

$O(1)$: $Y'' + 2Y' = 0$

$$\frac{Y''}{Y'} = -2$$

$$\ln Y' = -2X + a$$

$$Y' = ae^{-2X}$$

$$Y = ae^{-2X} + b$$

$$0 = Y(0) = a + b \Rightarrow b = -a$$

$$Y = a(e^{-2X} - 1)$$

MATCH:

$$\lim_{x \to 1^-} y(x) = \lim_{X \to \infty} Y(X)$$

$$e^{\tan^{-1} 1} = -a$$

$$-e^{\frac{\pi}{4}} = a$$

SOLUTION:

$$y(x) + Y\left(\frac{x-1}{\varepsilon}\right) - e^{\frac{\pi}{4}}$$

$$= e^{\tan^{-1} x} - e^{\frac{\pi}{4}}\left(e^{\frac{x-1}{\varepsilon}} - 1\right) - e^{\frac{\pi}{4}}$$

$$= e^{\tan^{-1} x} - e^{\frac{\pi}{4} + \frac{x-1}{\varepsilon}}$$

S'014 #5

$$\int^{1} e^{ixt^2} dt$$

lowest period @ $t=0$:

$$\sqrt{\frac{\pi}{2x}}\, e^{i(x\cdot 0^2 + \frac{\pi}{4})}$$

$$= \sqrt{\frac{\pi}{2x}}\, e^{i\frac{\pi}{4}}$$

S'014 #6

$$\frac{d}{dx}\left[c^2(x)\frac{dh}{dx}\right]+w^2h=0$$

For a good approximation, need $w \gg \|c\|_\infty$

$$c^2h''+cc'h'+w^2h=0$$

$$h=\exp(wu_0+u_1)$$

$$c^2\left[(wu_0''+u_1'')+(wu_0'+u_1')^2\right]$$
$$+cc'(wu_0'+u_1')+w^2=0$$

$O(w^2)$: $$c^2(u_0')^2+1=0$$
$$u_0'=\pm ic$$
$$u_0=A\pm i\int^x c(t)dt$$

$O(w)$: $$c^2u_0''+2c^2u_0'u_1'+cc'u_0'=0$$
$$\pm ic^2c'\pm 2ic^3u_1'\pm ic^2c'=0$$
$$c^3u_1'+c^2c'=0$$
$$u_1'=-\frac{c'}{c}$$
$$u_1=B+\ln c$$

$$h=\exp\left(Aw\pm iw\int^x c(t)dt+B+\ln c\right)$$
$$h(x)=c(x)\exp\left(Aw+B\pm iw\int^x c(t)dt\right)$$

as $c\to c_\pm$

$$h(x)\to c_\pm\exp(Aw+B)\exp(\pm iwc_\pm x)$$

$$\text{wavelengths}=\frac{2\pi}{wc_\pm}$$

$$h_-=\lim_{x\to-\infty}|h(x)|=c_-\exp(Aw+B) \Rightarrow \exp(Aw+B)=\frac{h_-}{c_-}$$

$$h_+=\lim_{x\to\infty}|h(x)|=c_+\exp(Aw+B)=\frac{c_+}{c_-}h_-$$