UC Davis

Applied Math

Prelim Solutions

"Back of the Napkin" style

Compiled 9/13/2016

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If you find a mistake, please send the correct solution to the author.

S'013 #1 (Part 1)

$$\dot{x} = -y + ax(x^2+y^2) - by(x^2+y^2)$$
$$\dot{y} = x + ay(x^2+y^2) + bx(x^2+y^2)$$

Note: $2x\dot{x} = -2xy + 2ax^2(x^2+y^2) - 2bxy(x^2+y^2)$

and $2y\dot{y} = 2xy + 2ay^2(x^2+y^2) + 2bxy(x^2+y^2)$

$$\therefore \quad 2x\dot{x} + 2y\dot{y} = 2a(x^2+y^2)^2$$

fixed pts: $2x(0) + 2y(0) = 2a(x^2+y^2)^2$

$$0 = 2a(x^2+y^2)^2$$

$\Rightarrow a = 0$ or $x^2+y^2 = 0$

$\therefore (x^*, y^*) = (0,0)$

(Consequently, we also see that $\frac{d}{dt}(x^2+y^2) = 2a(x^2+y^2)^2$

so $(0,0)$ is a source for $a>0$

and a sink for $a<0$)

Linear stability:

$$J(x,y) = \begin{bmatrix} 3ax^2 + ay^2 - 2bxy & -1 + 2axy - bx^2 - 3by^2 \\ 1 + 2axy + 3bx^2 + by^2 & ax^2 + 3ay^2 + 2bxy \end{bmatrix}$$

$$J(0,0) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\lambda = \pm i$$

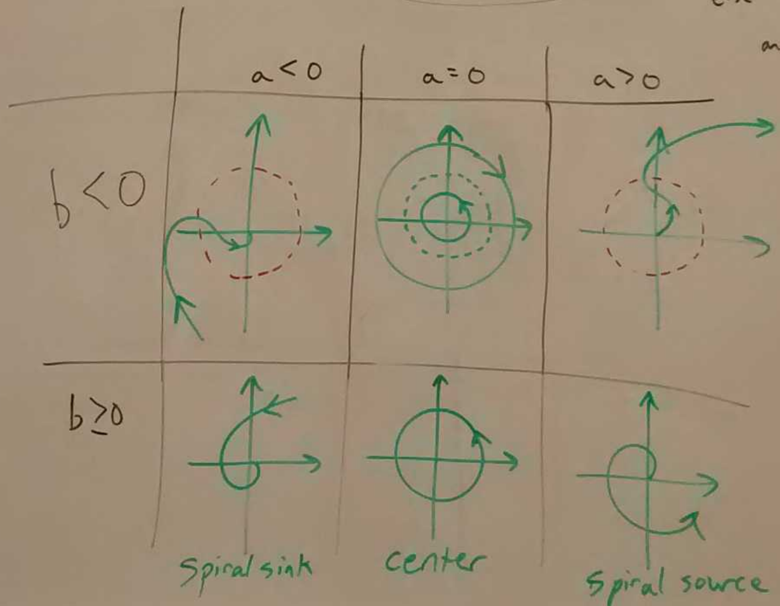
linearity predicts a center, but with the previous note, we know this is a spiral source for $a>0$ and a spiral sink for $a<0$.

S'013 #1 (Part 2)

$$\dot{x} = -y + ax(x^2+y^2) - by(x^2+y^2)$$
$$\dot{y} = x + ay(x^2+y^2) + bx(x^2+y^2)$$
$$r^2 = x^2+y^2$$
$$2r\dot{r} = 2x\dot{x} + 2y\dot{y}$$
$$= 2a(x^2+y^2)^2$$
$$= 2ar^4$$
$$\dot{r} = ar^3$$

$$x = r\cos\theta$$
$$\dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta$$
$$-y + ax(x^2+y^2) - by(x^2+y^2) = \dot{r}\cos\theta - r\dot{\theta}\sin\theta$$
$$-r\sin\theta + ar^3\cos\theta - br^3\sin\theta = ar^3\cos\theta - r\dot{\theta}\sin\theta$$
$$-r\sin\theta\left[1 + br^2\right] = -r\sin\theta\left[\dot{\theta}\right]$$
$$\dot{\theta} = 1 + br^2$$

Note: if $b \geq 0$, $\dot{\theta} > 0$
else $\dot{\theta} > 0$ if $r < \sqrt{\frac{1}{-b}}$
and $\dot{\theta} < 0$ if $r > \sqrt{\frac{1}{-b}}$



Stokes'
circulati
Laplace
$-\frac{\partial V}{\partial x} =$
Lyapunov
Poincaré-
Stirling
G
G
$L - \sum$
Practice Probl

S'013 #2

$$\frac{dx_n}{dt} = n^2 x_n^3 \qquad x_n(0) = c_n, \ n \in \mathbb{N}$$

$$\frac{1}{x_n^3}\dot{x}_n = n^2$$

$$-\frac{1}{2x_n^2} = n^2 t + C_1$$

$$x_n^2 = \frac{-1}{2n^2 t + C_2}$$

$$c_n^2 = \frac{-1}{C_2}$$

$$C_2 = -\frac{1}{c_n^2}$$

$$x_n^2 = \frac{-1}{2n^2 t - \frac{1}{c_n^2}}$$

$$x_n^2 = \frac{c_n^2}{1 - 2c_n^2 n^2 t}$$

$$x_n = \frac{c_n}{\sqrt{1 - 2c_n^2 n^2 t}}$$

A little fuzzy on this part

If $\sum_{n=1}^{\infty} n^2 c_n^2 \leq 1$,

$$\sum_{n=1}^{\infty} x_n^2 = \sum_{n=1}^{\infty} \frac{c_n^2}{1 - 2c_n^2 n^2 t} = \sum_{n=1}^{\infty} \frac{n^2 c_n^2}{n^2 - 2c_n^2 n^4 t} \overset{*}{\leq} \sum_{n=1}^{\infty} \frac{1}{\varepsilon} n^2 c_n^2 \leq \frac{1}{\varepsilon} < \infty$$

* since $|x_n| < \infty$, we must have $\left(c_n = 0 \text{ or } \exists M > 0 \text{ s.t. } n^2 - 2c_n^2 n^4 t \geq \varepsilon\right) \forall n$

$$n^2 - \varepsilon \geq 2c_n^2 n^4 t$$

$$t \leq \frac{n^2 - \varepsilon}{2c_n^2 n^4} \sim \frac{1}{2c_n^2 n^2}$$

ideally, $T := \inf\left\{ \frac{1}{2c_n^2 n^2} \,\middle|\, n \in \mathbb{N} \right\}$

But since $\sum_{n=1}^{\infty} n^2 c_n^2 \leq 1 \Rightarrow n^2 c_n^2 \leq 1 \ \forall n$,

can use $T = \frac{1}{2}$ (more conservative estimate)

If $\sum_{n=1}^{\infty} c_n^2 = 1$, then $T := \inf\left\{ \frac{1}{2n^2 c_n^2} \right\}$ can be arbitrarily small by choice of $c_n = \delta_{nN}$ for N large.

Not thrilled with this solution

S'013 #3 (Part 1)

$$Lu = -(pu')' + qu = f \qquad u(a) = u'(b) = 0$$

Let u_1, u_2 be independent solutions to $Lu = 0$
with $u_1(a) = u_2'(b) = 0$
and $u_1'(b) \neq 0$, $u_2(a) \neq 0$.

want $G(x,\xi)$ s.t. $-(pG_x)_x + qG = \delta(x-\xi)$ and $G(a,\xi) = G_x(b,\xi) = 0$

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for $x \neq \xi$ $\quad -(pG_x)_x + qG = 0$

$\therefore G(x,\xi) = A_1(\xi)u_1(x) + A_2(\xi)u_2(x)$, $x \neq \xi$

$$\therefore G(x,\xi) = \begin{cases} A_1(\xi)u_1(x) + A_2(\xi)u_2(x) & x<\xi \\ B_1(\xi)u_1(x) + B_2(\xi)u_2(x) & x>\xi \end{cases}$$

BCs:

$$0 = G(a,\xi) = A_1(\xi)u_1(a) + A_2(\xi)u_2(a)$$
$$= A_2(\xi)u_2(a)$$
$$u_2(a) \neq 0 \Rightarrow A_2(\xi) = 0$$
$$0 = G_x(b,\xi) = B_1(\xi)u_1'(b) + B_2(\xi)u_2'(b)$$
$$= B_1(\xi)u_1'(b)$$
$$u_1'(b) \neq 0 \Rightarrow B_1(\xi) = 0.$$

so far

$$G(x,\xi) = \begin{cases} A_1(\xi)u_1(x) & x<\xi \\ B_2(\xi)u_2(x) & x>\xi \end{cases}$$

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continuity: $\quad A_1(\xi)u_1(\xi) = B_2(\xi)u_2(\xi)$

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unit impulse: $\quad -(pG_x)_x + qG = \delta(x-\xi)$

$$\int_{\xi^-}^{\xi^+} \left[(-pG_x)_x + qG\right] dx = 1$$

$$\left[-pG_x\right]_{\xi^-}^{\xi^+} + \int_{\xi^-}^{\xi^+} qG\, dx = 1$$

$$-p(\xi)\left[G_x(\xi^+) - G_x(\xi^-)\right] + 0 = 1$$

$$-p(\xi)\left[B_2(\xi)u_2'(\xi) - A_1(\xi)u_1'(\xi)\right] = 1$$

$$-p(\xi)\left[B_2(\xi)u_1(\xi)u_2'(\xi) - A_1(\xi)u_1(\xi)u_1'(\xi)\right] = u_1(\xi)$$

$$-p(\xi)\left[B_2(\xi)u_1(\xi)u_2'(\xi) - B_2(\xi)u_2(\xi)u_1'(\xi)\right] = u_1(\xi)$$

$$-p(\xi)B_2(\xi)W(\xi) = u_1(\xi)$$

$$\therefore B_2(\xi) = -\frac{u_1(\xi)}{p(\xi)W(\xi)}$$

$$-p(\xi)\left[B_2(\xi)u_2(\xi)u_2'(\xi) - A_1(\xi)u_2(\xi)u_1'(\xi)\right] = u_2(\xi)$$

$$-p(\xi)\left[A_1(\xi)u_1(\xi)u_2'(\xi) - A_1(\xi)u_2(\xi)u_1'(\xi)\right] = u_2(\xi)$$

$$-p(\xi)A_1(\xi)W(\xi) = u_2(\xi)$$

$$\therefore A_1(\xi) = -\frac{u_2(\xi)}{p(\xi)W(\xi)}$$

$$\therefore G(x,\xi) = \begin{cases} -\dfrac{u_1(x)u_2(\xi)}{p(\xi)W(\xi)} & x<\xi \\ -\dfrac{u_1(\xi)u_2(x)}{p(\xi)W(\xi)} & x>\xi \end{cases}$$

S'013 #3 (Part 2)

$$Lu = -(pu')' + qu = f \qquad u(a) = u'(b) = 0$$

Claim: pW is constant.

pf/ $\frac{d}{dx}(pW) = \frac{d}{dx}\left(p u_2' u_1 - p u_1' u_2\right)$

$= (pu_2')' u_1 + p u_2' u_1' - (pu_1')' u_2 - p u_1' u_2'$

$= (pu_2')' u_1 - (pu_1')' u_2$

$= q u_2 u_1 - q u_1 u_2$

$= 0$ //

$$G(x,\xi) = \begin{cases} -\dfrac{u_1(x)u_2(\xi)}{p(\xi)W(\xi)} & x < \xi \\ -\dfrac{u_1(\xi)u_2(x)}{p(\xi)W(\xi)} & x > \xi \end{cases}$$

$$G(\xi,x) = \begin{cases} -\dfrac{u_1(\xi)u_2(x)}{p(x)W(x)} & \xi < x \\ -\dfrac{u_1(x)u_2(\xi)}{p(x)W(x)} & \xi > x \end{cases}$$

$$= \begin{cases} -\dfrac{u_1(x)u_2(\xi)}{p(x)W(x)} & x < \xi \\ -\dfrac{u_1(\xi)u_2(x)}{p(x)W(x)} & x > \xi \end{cases}$$

$$\overset{pW\text{ const}}{=} \begin{cases} -\dfrac{u_1(x)u_2(\xi)}{p(\xi)W(\xi)} & x < \xi \\ -\dfrac{u_1(\xi)u_2(x)}{p(\xi)W(\xi)} & x > \xi \end{cases}$$

$$= G(x,\xi)$$

S'013 #3 (Part 3)

$$G(x,\xi) = G(\xi,x)$$

The response as measured at x
due to a unit impulse at ξ

is the same as

the response as measured at ξ
due to a unit impulse at x

S'013 #4 Page 1

$u \in C^2[1,2]$ $u(1)=0$ $u(2)=1$

$$J(u) = \int_1^2 \frac{\sqrt{1+(u')^2}}{x}\,dx$$

E-L: $\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0$

$$0 - \frac{d}{dx}\frac{u'}{x\sqrt{1+(u')^2}} = 0$$

$$\frac{u'}{x\sqrt{1+(u')^2}} = C$$

$$(u')^2 = C^2x^2(1+(u')^2)$$

$$(1-C^2x^2)(u')^2 = C^2x^2$$

$$u' = \pm\sqrt{\frac{C^2x^2}{1-C^2x^2}} = \frac{Cx}{\sqrt{1-C^2x^2}}$$

BCs induce positive root

$$u = \int \frac{Cx}{\sqrt{1-C^2x^2}}\,dx$$

Let $y := 1-C^2x^2$

$$dy = -2C^2x\,dx \Rightarrow -\frac{dy}{2C} = Cx\,dx$$

$$u = -\frac{1}{2C}\int y^{-\frac{1}{2}}\,dy$$

$$u = -\frac{1}{C}y^{\frac{1}{2}} + D$$

$$u = -\frac{1}{C}\sqrt{1-C^2x^2} + D$$

$0 = u(1) = -\frac{1}{C}\sqrt{1-C^2} + D$ $\qquad$ $1 = u(2) = -\frac{1}{C}\sqrt{1-4C^2} + D$

$\sqrt{1-C^2} = CD$ $\qquad$ $CD - C = \sqrt{1-4C^2}$

S'013 #4 Page 2

$\sqrt{1-C^2} = CD$ $\qquad\qquad$ $CD - C = \sqrt{1-4C^2}$

$$\sqrt{1-C^2} - C = \sqrt{1-4C^2}$$

$$1 - C^2 - 2C\sqrt{1-C^2} + C^2 = 1 - 4C^2$$

$$-2C\sqrt{1-C^2} = -4C^2$$

$$\sqrt{1-C^2} = 2C$$

$$1 - C^2 = 4C^2$$

$$1 = 5C^2$$

$$\frac{1}{\sqrt{5}} = C$$

$$CD = \sqrt{1-C^2}$$

$$\frac{1}{\sqrt{5}} D = \sqrt{1-\frac{1}{5}}$$

$$D = \sqrt{5-1}$$

$$D = 2$$

$$u = -\sqrt{5}\sqrt{1-\frac{1}{5}x^2} + 2$$

$$= -\sqrt{5-x^2} + 2$$

S'013 #5

$$u_{tt} - c^2 u_{xx} = A\sin\left(\frac{\pi x}{L}\right)\sin\omega t \qquad u(0,t) = u(L,t) = 0 \qquad A \neq 0$$
$$u(x,0) = u_t(x,0) = 0$$

$$u(x,t) = \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} A\sin\left(\frac{\pi y}{L}\right)\sin\omega s \, dy\, ds$$

$$= \frac{A}{2c}\int_0^t \sin\omega s\left[-\frac{L}{\pi}\cos\frac{\pi y}{L}\right]_{x-c(t-s)}^{x+c[t-s]} ds$$

$$= -\frac{AL}{2\pi c}\int_0^t \sin\omega s\left[\cos\left(x+c[t-s]\right) - \cos\left(x-c[t-s]\right)\right] ds$$

= solution

S'013 #6 (Part 1)

$$u_t + u u_x = \nu u_{xx}$$

$$\mathcal{U} \xleftarrow{x} u(x,t) \xrightarrow{x} 0$$

editorial: too many symbols look like 'u'

Dimensions: $\dfrac{\frac{L}{T}}{T} + \frac{L}{T}\left(\dfrac{\frac{L}{T}}{L}\right) = \nu\left(\dfrac{\frac{L}{T}}{L^2}\right)$

$$\frac{L}{T^2} + \frac{L}{T^2} = \nu\left(\frac{1}{LT}\right)$$

$$\frac{L^2}{T} = \nu$$

ν has dimensions $\frac{\text{length}^2}{\text{time}}$

$\mathcal{U}$ has dimensions $\frac{\text{length}}{\text{time}}$

$$\tau := \frac{t}{T} \qquad X := \frac{x}{L} \qquad v(X,\tau) := \frac{u(x,t)}{\mathcal{U}}$$

$$\frac{\partial u}{\partial v}\frac{\partial v}{\partial \tau}\frac{\partial \tau}{\partial t} + u\frac{\partial u}{\partial v}\frac{\partial v}{\partial X}\frac{\partial X}{\partial x} = \nu\frac{\partial}{\partial X}\left(\frac{\partial X}{\partial x}\frac{\partial u}{\partial v}\frac{\partial v}{\partial X}\frac{\partial X}{\partial x}\right)$$

$$\mathcal{U} v_\tau \frac{1}{T} + \mathcal{U} v\, \mathcal{U} v_X \frac{1}{L} = \nu\frac{\partial}{\partial X}\left[\frac{1}{L}\mathcal{U} v_X \frac{1}{L}\right]$$

$$\frac{\mathcal{U}}{T} v_\tau + \frac{\mathcal{U}^2}{L} v v_X = \frac{\mathcal{U}\nu}{L^2} v_{XX}$$

$$v_\tau + \frac{\mathcal{U}T}{L} v v_X = \frac{\nu T}{L^2} v_{XX}$$

$$\frac{L}{T} := \mathcal{U} \qquad \frac{L^2}{T} := \nu$$

$$\therefore L = \frac{\nu}{\mathcal{U}}, \quad T = \frac{\nu}{\mathcal{U}^2}$$

$$\boxed{v_\tau + v v_X = v_{XX}, \quad 1 \xleftarrow{X} v(X,\tau) \xrightarrow{X} 0}$$

S'013 #6 (Part 2)

$$u_t + uu_x = \nu u_{xx}$$

$$\mathcal{U} \xleftarrow{x} u(x,t) \xrightarrow{x} 0$$

Travelling waves: $u = u(x-ct)$

The typical speed should be on the order of magnitude $\mathcal{U}$

and the typical length should be on the order of $\dfrac{\nu}{\mathcal{U}}$

$$-cu' + uu' = \nu u''$$

still need part (c)