UC Davis

Applied Math

Prelim Solutions

"Back of the Napkin" style

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If you find a mistake,
please send the correct
solution to the author.

F'015 #1

$$\dot{x} = -y - x^3$$
$$\dot{y} = x^5$$

<u>linear stability:</u>

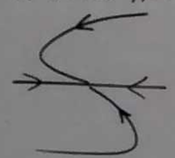
$$J(x,y) = \begin{bmatrix} -3x^2 & -1 \\ 5x^4 & 0 \end{bmatrix}$$

$$J(0,0) = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$$

$\lambda = 0, 0$

Frankly, this is not a strong enough linear result to conclude much of anything.

The repeated eigenvalue and lone eigenvector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ suggest a twist in the flow



$$V := Ax^6 + By^2$$

$$\begin{aligned} \dot{V} &= 6Ax^5\dot{x} + 2By\dot{y} \\ &= 6Ax^5(-y-x^3) + 2By(x^5) \\ &= -6Ax^5y + 2Bx^5y - 6Ax^8 \\ &= -2(3A-B)x^5y - 6Ax^8 \end{aligned}$$

Let $(A, B) = (1, 3)$

so $V = x^6 + 3y^2$

Then $\dot{V} = -6x^8 < 0$ away from $(0,0)$

$\therefore$ the origin is an attractor

F'OIS #2

$$x_{n+1} = -\mu x_n - x_n^3$$

Fixed pts: $x_n = -\mu x_n - x_n^3$

$$x_n^3 + (\mu+1)x_n = 0$$

$$x_n(x_n^2 + \mu + 1) = 0$$

$x_n = 0$ or $x_n^2 = -\mu - 1$ $\mu < -1$

$$x_n = \sqrt{-\mu-1}$$

$$\frac{d}{dx_n}(x_{n+1} - x_n) = \frac{d}{dx_n}\left[(-\mu-1)x_n - x_n^3\right]$$

$$= -\mu - 1 - 3x_n^2$$

stable if $\mu > -1 - 3(x^*)^2$

In particular, $x^* = 0$ is stable for $\mu > -1$

There will be a flip bifurcation as μ passes through 1.

$$x_{n+2} = -\mu(-\mu x_n - x_n^3) - (-\mu x_n - x_n^3)^3$$

$$= \mu^2 x_n + \mu x_n^3 + \mu^3 x_n^3 + O(x_n^5)$$

$$= (1 + 2\varepsilon + \varepsilon^2)x_n + (1+\varepsilon)x_n^3 + (1 + 3\varepsilon + 3\varepsilon^2 + \varepsilon^3)x_n^3 + O(x_n^5)$$

$$= (1 + 2\varepsilon)x_n + 2x_n^3 + O(\varepsilon^2) + O(\varepsilon x_n^3) + O(x_n^5)$$

$$\approx (1 + 2\varepsilon)x_n + 2x_n^3$$

this is a subcritical flip bifurcation.

F'015 #3 (Page 1)

curve of length ℓ in upper half-plane passing through $(-a,0)$ and $(a,0)$ enclosing largest area.

$$\max \ J(y) = \int_{-a}^{a} y \, dx \qquad \text{s.t.} \quad K(y) = \int_{-a}^{a} \sqrt{1+(y')^2}\, dx = \ell$$

Construct Lagrangian $L = J + \lambda K = \int_{-a}^{a} \left(y + \lambda\sqrt{1+(y')^2} \right) dx$

E-L: $$\frac{\partial L}{\partial y} - \frac{d}{dx}\frac{\partial L}{\partial y'} = 0$$

$$1 - \frac{d}{dx}\frac{\lambda y'}{\sqrt{1+(y')^2}} = 0$$

$$\frac{\lambda y'}{\sqrt{1+(y')^2}} = x + C$$

$$\lambda^2 (y')^2 = (x+C)^2(1+(y')^2)$$

$$(\lambda^2 - [x+C]^2)(y')^2 = [x+C]^2$$

$$y' = \frac{x+C}{\sqrt{\lambda^2 - [x+C]^2}}$$

$$y = \int \frac{x+C}{\sqrt{\lambda^2-[x+C]^2}}\, dx$$

$$z := \lambda^2 - [x+C]^2$$

$$dz = -2[x+C]\, dx$$

$$y = -\frac{1}{2}\int \frac{1}{\sqrt{z}}\, dz$$

$$y = -\sqrt{z} + D$$

$$y = -\sqrt{\lambda^2 - [x+C]^2} + D$$

F'015 #3 (Page 2)

$$y = -\sqrt{\lambda^2 - [x+C]^2} + D$$

$$0 = y(-a) = -\sqrt{\lambda^2 - [C-a]^2} + D \qquad 0 = y(a) = -\sqrt{\lambda^2 - [C+a]^2} + D$$

$$\lambda^2 - [C-a]^2 = D^2 \qquad \lambda^2 - [C+a]^2 = D^2$$

$$[C+a]^2 - [C-a]^2 = 0$$

$$4aC = 0$$

$$C = 0$$

$$\lambda^2 + a^2 = D^2$$

Recall $$y' = \frac{x+C}{\sqrt{\lambda^2 - [x+C]^2}} = \frac{x}{\sqrt{\lambda^2 - x^2}}$$

$$\ell = \int_{-a}^{a} \sqrt{1+(y')^2}\,dx = \int_{-a}^{a} \sqrt{1 + \frac{x^2}{\lambda^2 - x^2}}\,dx = \int_{-a}^{a} \frac{\lambda}{\sqrt{\lambda^2 - x^2}}\,dx = 2\int_{0}^{a} \frac{\lambda}{\sqrt{\lambda^2 - x^2}}\,dx$$

Let $x = \lambda \sin\theta$

$dx = \lambda \cos\theta\, d\theta$

$$\ell = 2\int_{0}^{\theta(a)} \frac{\lambda^2 \cos\theta\, d\theta}{\sqrt{\lambda^2(1-\sin^2\theta)}} = 2\int_{0}^{\theta(a)} \lambda\, d\theta = 2\lambda\theta\Big|_{0}^{\theta(a)} = 2\lambda\theta(a)$$

$$\ell = 2\lambda\theta_0 \qquad \lambda \sin\theta_0 = a$$

$$y = -\sqrt{\lambda^2 - x^2} + \sqrt{\lambda^2 + a^2}$$

$$x^2 + \left(y - \sqrt{\lambda^2 + a^2}\right)^2 = \lambda^2$$

$$\boxed{x^2 + \left(y - \sqrt{\left(\frac{\ell}{2\theta_0}\right)^2 + a^2}\right)^2 = \left(\frac{\ell}{2\theta_0}\right)^2 , \quad \sin\theta_0 = \frac{2a\theta_0}{\ell}}$$

it is a portion of a circle

F'015 #4

$$u_t = u_{xx} + u_{yy} \qquad a > b$$

$$u_x(0,y,t) = u_x(a,y,t) = 0$$

$$u_y(x,0,t) = u_y(x,b,t) = 0$$

$$u(x,y,0) = f(x,y)$$

$$u(x,y,t) = X(x)\,Y(y)\,T(t)$$

$$XYT' = X''YT + XY''T$$

$$\frac{T'}{T} = \frac{X''}{X} + \frac{Y''}{Y} := -\frac{\pi^2 m^2}{a^2} - \frac{\pi^2 n^2}{b^2}$$

wlog $$T' = e^{-\pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)t}$$

$$X = \sqrt{\frac{2-\delta_{m0}}{a}} \cos \frac{\pi m x}{a}$$

$$Y = \sqrt{\frac{2-\delta_{n0}}{b}} \cos \frac{\pi n y}{b}$$

$$u(x,y,t) = \sum_{m,n=0}^{\infty} \sqrt{\frac{(2-\delta_{n0})(2-\delta_{m0})}{ab}}\, C_{mn}\, e^{-\pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)t} \cos \frac{\pi m x}{a} \cos \frac{\pi n y}{b}$$

$$C_{mn} = \sqrt{\frac{(2-\delta_{n0})(2-\delta_{m0})}{ab}} \int_0^a \int_0^b f(x,y) \cos \frac{\pi m x}{a} \cos \frac{\pi n y}{b}\, dy\, dx$$

if $f \equiv c$, $$C_{mn} = \frac{1}{\sqrt{ab}}\, abc = c\sqrt{ab}$$ if $(m,n) = (0,0)$, else 0

$$u(x,y,t) = \frac{1}{\sqrt{ab}}\, c\sqrt{ab}\, e^{-\pi^2(0)t}$$

$$= c$$

F'015 #5 (Page 1)

$f'' + w^2 f = g \qquad f'(0) = f'(1) = 0 \qquad w \neq \pi k$ for any $k \in \mathbb{Z}$

Consider $f'' + w^2 f = 0$, solved by $f = c_1 \cos wx + c_2 \sin wx$

find u_1

$f'(x) = -c_1 w \sin wx + c_2 w \cos wx$

$0 = f'(0) = c_2 w \Rightarrow c_2 = 0$

$\therefore$ $w \cos wx$ is a solution satisfying $f'(0) = 0$

find u_2

$f'(x) = -c_3 w \sin wx + c_4 w \cos wx$

$0 = f'(1) = -c_3 w \sin w + c_4 w \cos w$

$c_3 \sin w = c_4 \cos w \qquad c_3 := \cos w \Rightarrow c_4 = \sin w$

$\therefore$ $\cos w \cos wx + \sin w \sin wx$ is a solution satisfying $f'(1) = 0$

$= \cos[w(1-x)]$

$u_1(x) = w \cos wx \qquad u_2(x) = \cos[w(1-x)]$

$$W(x) = \begin{vmatrix} w \cos wx & \cos[w(1-x)] \\ -w^2 \sin wx & w \sin[w(1-x)] \end{vmatrix} = w^2 \sin[w(1-x)] \cos wx + w^2 \cos[w(1-x)] \sin wx$$

$$= w^2 \sin w$$

$$G(x,\xi) = \begin{cases} \dfrac{\cos wx \cos[w(1-\xi)]}{w \sin w} & x < \xi \\[2ex] \dfrac{\cos w\xi \cos[w(1-x)]}{w \sin w} & x > \xi \end{cases}$$

If $w = 0$, then constant functions satisfy the homogeneous problem <u>and</u> both BCs, so we must enforce a solvability condition on g.

$$0 = \int_0^1 g(x)\,dx$$

in which case we must use the generalized Green's function (Next Page)

F'015 #5 (Page 2)

$$f'' = g \qquad f'(0) = f'(1) = 0 \qquad 0 = \int_0^1 g(x)\,dx$$

Subtract the projection onto constant function 1

$$f''(x) = g(x) - 1\int_0^1 1\,g(\xi)\,d\xi = g(x) - \int_0^1 g(\xi)\,d\xi$$

$$\int_0^1 G_{xx}\,g(\xi)\,d\xi = \int_0^1 \delta(x-\xi)\,g(\xi)\,d\xi - \int_0^1 g(\xi)\,d\xi$$

$$G_{xx} = \delta(x-\xi) - 1$$

$$G = \begin{cases} -\frac{1}{2}x^2 + a_1 x + a_2 & x < \xi \\ -\frac{1}{2}x^2 + b_1 x + b_2 & x > \xi \end{cases}$$

$$0 = G_x(0,\xi) = a_1$$

$$0 = G_x(1,\xi) = -1 + b_1 \Rightarrow b_1 = 1$$

$$-\frac{1}{2}\xi^2 + a_2 = -\frac{1}{2}\xi^2 + \xi + b_2 \Rightarrow a_2 = \xi + b_2$$

$$G = \begin{cases} -\frac{1}{2}x^2 + \xi + b_2 & x < \xi \\ -\frac{1}{2}x^2 + x + b_2 & x > \xi \end{cases}$$

$$0 = \int_0^1 1G\,dx = \int_0^1 \left(-\frac{1}{2}x^2 + b_2\right)dx + \int_0^\xi \xi\,dx + \int_\xi^1 x\,dx$$

$$= -\frac{1}{6} + b_2 + \xi^2 + \frac{1}{2} - \frac{1}{2}\xi^2$$

$$= b_2 + \frac{1}{2}\xi^2 + \frac{1}{3} \qquad \therefore\ b_2 = -\frac{1}{2}\xi^2 - \frac{1}{3}$$

$$G(x,\xi) = \begin{cases} -\frac{1}{2}x^2 + \xi - \frac{1}{2}\xi^2 - \frac{1}{3} & x < \xi \\ -\frac{1}{2}x^2 + x - \frac{1}{2}\xi^2 - \frac{1}{3} & x > \xi \end{cases}$$

F'015 #6

$$\varepsilon y'' + y(y'+3) = 0$$
$$y(0) = -1 \quad y(1) = 1$$

The provided sketch suggests a layer at $x=0.5$

outer solution:

$O(1)$: $\quad y(y'+3)=0$

$y=0$ (trivial) or $y'+3=0$

$$y = -3x + c$$

on the left side,

$$-1 = y_L(0) = -3(0) + c_L \Rightarrow c_L = -1$$

on the right side,

$$1 = y_R(1) = -3(1) + c_R \Rightarrow c_R = 4$$

$\therefore\ y_L(x) = -3x - 1$

& $y_R(x) = -3x + 4$

Note: the picture also supports this.

inner solution:

$$x := \frac{1}{2} + \varepsilon^{\alpha} X \qquad Y(X) := y(x)$$

$$\varepsilon^{1-2\alpha} Y'' + Y(\varepsilon^{-\alpha} Y' + 3) = 0$$

$$1-2\alpha = -\alpha \Rightarrow \alpha = 1$$

$$\varepsilon^{-1} Y'' + \varepsilon^{-1} Y Y' + 3Y = 0$$

$$Y'' + YY' + 3\varepsilon Y = 0$$

$O(1)$: $\quad Y'' + YY' = 0$

$$Y' + \frac{1}{2}Y^2 = a$$

$$Y' = a - \frac{1}{2}Y^2$$

$$\frac{Y'}{a - \frac{1}{2}Y^2} = 1$$

$$\sqrt{\frac{2}{a}} \tanh^{-1}\left(Y\sqrt{\frac{1}{2a}}\right) = X + b$$

$$\tanh^{-1}\left(Y\frac{1}{\sqrt{2a}}\right) = \sqrt{\frac{a}{2}}X + b$$

$$\frac{1}{\sqrt{2a}} Y = \tanh\left(\sqrt{\frac{a}{2}}X + b\right)$$

$$Y = \sqrt{2a}\tanh\left(\sqrt{\frac{a}{2}}X + b\right)$$

matching:

$0 = Y(0)$ from graph (can also match Y_L to Y_R @ 0.5)

$= \sqrt{2a}\tanh(b) \Rightarrow b = 0$

right side

$$\lim_{x\to\frac{1}{2}^+} y_R(x) = \lim_{X\to\infty} Y_R(X)$$

$$-\frac{3}{2} + 4 = \sqrt{2a_R}$$

$$\frac{5}{2} = \sqrt{2a_R}$$

$$\frac{25}{8} = a_R$$

left side

$$\lim_{x\to\frac{1}{2}^-} y_L(x) = \lim_{X\to-\infty} Y_L(X)$$

$$-\frac{3}{2} - 1 = -\sqrt{2a_L}$$

$$-\frac{5}{2} = -\sqrt{2a_L}$$

$$\frac{25}{8} = a_L$$

SOLUTION

$$\begin{cases} y_L(x) + Y_L\left(\frac{x-\frac{1}{2}}{\varepsilon}\right) + \frac{5}{2} & x < 0.5 \\ y_R(x) + Y_R\left(\frac{x-\frac{1}{2}}{\varepsilon}\right) - \frac{5}{2} & x \geq 0.5 \end{cases}$$

$$= \begin{cases} -3x + \frac{3}{2} + \frac{5}{2}\tanh \frac{5}{4}\left(\frac{x-\frac{1}{2}}{\varepsilon}\right) & x < 0.5 \\ -3x + \frac{3}{2} + \frac{5}{2}\tanh \frac{5}{4}\left(\frac{x-\frac{1}{2}}{\varepsilon}\right) & x \geq 0.5 \end{cases}$$

$$= \boxed{-3x + \frac{3}{2} + \frac{5}{2}\tanh \frac{5}{4}\left(\frac{x-\frac{1}{2}}{\varepsilon}\right)}$$