UC Davis
Applied Math
Prelim Solutions
"Back of the Napkin" style

Compiled 9/13/2016
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If you find a mistake,
please send the correct
solution to the author.

F'014 #1

$\theta = 0$: $\frac{dx}{dt} = \left(\frac{1}{\sqrt{h^2+x^2}} - 1\right)x$

fixed pts: $0 = \left(\frac{1}{\sqrt{h^2+x^2}} - 1\right)x \Rightarrow \boxed{x=0}$ or

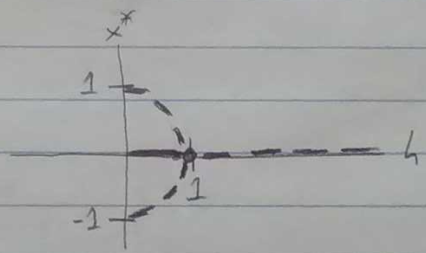
$h^2+x^2 = 1 \Rightarrow \boxed{x = \pm\sqrt{1-h^2}}$

stability: $\frac{\partial \dot{x}}{\partial x} = \left(\frac{1}{\sqrt{h^2+x^2}} - 1\right) + x\left(\frac{x}{(h^2+x^2)^{3/2}}\right)$

$= \frac{1}{\sqrt{h^2+x^2}} - 1 + \frac{x^2}{(h^2+x^2)^{3/2}}$

if $x=0$: $= -1 + \frac{1}{\sqrt{h^2}}$ (neg for small h)

(pos for large h)



$h = 1+\varepsilon$: $0 = \left(\frac{1}{\sqrt{h^2+2xh\sin\theta+x^2}} - 1\right)(x + h\sin\theta)$

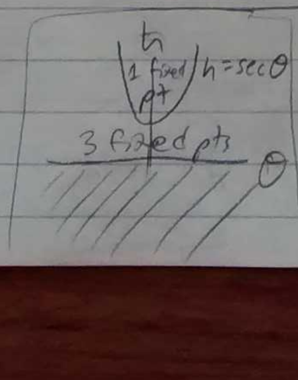
$\Rightarrow x = -h\sin\theta \approx -\sin\theta$ or

$x^2+h^2+2xh\sin\theta = 1$

$x^2+2xh\sin\theta+h^2-1=0$

$x = \frac{-2h\sin\theta \pm \sqrt{4h^2\sin^2\theta - 4h^2+4}}{2}$

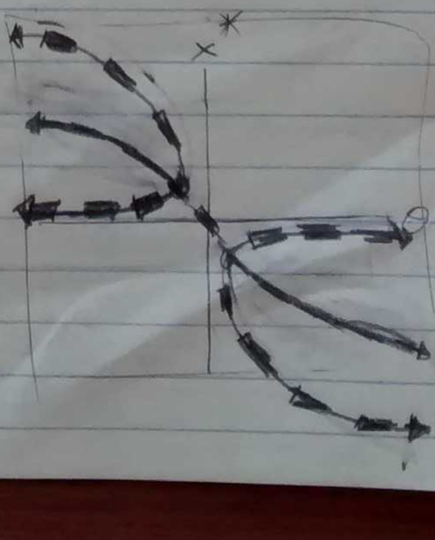
$= \frac{-2h\sin\theta \pm \sqrt{4-4h^2\cos^2\theta}}{2}$



$1 = h^2\cos^2\theta$

$\left|\frac{1}{h}\right| = |\cos\theta|$

$|h| = |\sec\theta|$



F'014 #2

$\dot{x}_1 = x_2$

$\dot{x}_2 = -g(x_1)$

$$\boxed{V(x_1) = \int_0^{x_1} g(x_1)}$$

A fun abuse of notation

has a local min @ x_1^*

$g \in C^1$

Show stable eq at $(x_1^*, 0)$

$0 = V'(x_1) = g(x_1)$

$0 > V''(x_1) = g'(x_1)$

Claim: $E = \frac{1}{2} x_2^2 + V(x_1)$ is a conserved quantity

pf/ $\dot{E} = x_2 \dot{x}_2 + V'(x_1)\dot{x}_1$

$= -x_2 g(x_1) + g(x_1) x_2$

$= 0$ //

Taylor expand E wrt x_1:

$$E = \frac{1}{2} x_2^2 + V(x_1^*) + V'(x_1^*)(x_1 - x_1^*) + \frac{V''(\xi)}{2}(x_1 - x_1^*)^2 \qquad |\xi - x_1^*| < \varepsilon$$

$$E = \frac{1}{2} x_2^2 + V(x_1^*) + 0 + \frac{V''(\xi)}{2}(x_1 - x_1^*)^2$$

$$2E - 2V(x_1^*) = x_2^2 + V''(x_1^*)(x_1 - x_1^*)^2$$

x_1^* isolated, local min $\Rightarrow V''(x_1^*) > 0$. ξ nearby $\Rightarrow V''(\xi) > 0$

$2E - 2V(x_1^*)$ constant

$\therefore$ these trajectories are ellipses w/ center $(x_1^*, 0)$

$\therefore$ nearby trajectories remain close

$\therefore$ Lyapnov stable

F'014 #3 (Page 1)

Brachistocrone problem (x_1, y_1) to (x_2, y_2)

$$\frac{1}{2}mv^2 + mgy = mgy_1$$

$$v^2 + 2gy = 2gy_1$$

$$v^2 = 2g(y_1 - y)$$

$$\min \quad J(y) = \int_{x_1}^{x_2} \frac{\sqrt{1+(y')^2}}{\sqrt{2g(y_1-y)}}\,dx$$

no explizit x dependence. $L - y'\frac{\partial L}{\partial y'} = C$

$$\sqrt{\frac{1+(y')^2}{2g(y_1-y)}} - \frac{(y')^2}{\sqrt{2g(y_1-y)}\sqrt{1+(y')^2}} = C$$

$$1+(y')^2 - (y')^2 = C\sqrt{2g(y_1-y)}\sqrt{1+(y')^2}$$

$$\frac{1}{C} = \sqrt{2g(y_1-y)}\sqrt{1+(y')^2}$$

$$\frac{1}{C^2} = 2g(y_1-y)(1+(y')^2)$$

$$\frac{1}{2gC^2(y_1-y)} = 1+(y')^2$$

$$\frac{1}{2gC^2(y_1-y)} - 1 = (y')^2$$

$$\sqrt{\frac{1-2gC^2(y_1-y)}{2gC^2(y_1-y)}} = y'$$

$$dx = \sqrt{\frac{2gC^2(y_1-y)}{1-2gC^2(y_1-y)}}\,y'dx$$

$$x = \int\sqrt{\frac{2gC^2(y_1-y)}{1-2gC^2(y_1-y)}}\,dy$$

F'014 #3 (Page 2)

let $2gC^2(y_1-y) = \sin^2\theta \qquad -2gC^2 dy = 2\sin\theta\cos\theta d\theta$

$$x = \frac{-1}{gC^2}\int\sqrt{\frac{\sin^2\theta}{1-\sin^2\theta}}\sin\theta\cos\theta d\theta$$

$$x = \frac{-1}{gC^2}\int \sin^2\theta d\theta$$

$$x = \frac{-1}{2gC^2}\int (1-\cos 2\theta)d\theta$$

$$x = \frac{-1}{2gC^2}\left[\theta - \tfrac{1}{2}\sin 2\theta\right] + D$$

Recall $2gC^2(y_1-y) = \sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$

$$-4gC^2 y = -4gC^2 y_1 + 1 - \cos 2\theta$$

$$\begin{cases} y = \dfrac{-1}{4gC^2}(1-\cos 2\theta) + y_1 \\ x = \dfrac{-1}{4gC^2}(2\theta - \sin 2\theta) + x_1 \end{cases}$$

$\alpha := -\dfrac{1}{4gC^2}$

$$\begin{cases} y = \alpha(1-\cos 2\theta) + y_1 \\ x = \alpha(2\theta - \sin 2\theta) + x_1 \end{cases} \qquad \theta \in [0, \tfrac{\pi}{4}]$$

use BC

$$\begin{cases} y = (y_2 - y_1)(1-\cos 2\theta) + y_1 \\ x = (x_2 - x_1)(2\theta - \sin 2\theta) + x_1 \end{cases} \qquad \theta \in [0, \tfrac{\pi}{4}]$$

F'014 #4

$x^2 y'' - xy' - 3y = x - 3$

$y(1) = y(2) = 0$

std form: $y'' - \frac{1}{x}y' - \frac{3}{x^2}y = \frac{1}{x} - \frac{3}{x^2} =: f(x)$

$y := x^n$ Consider homog case:

$n(n-1) - n - 3 = 0$

$n^2 - 2n - 3 = 0$

$n = -1, 3$

$y_1(x) = \frac{1}{x}, \quad y_2(x) = x^3$

$$y_p(x) = \left(-\int_1^x \frac{y_2(\xi)}{W(\xi)} f(\xi)\,d\xi\right) y_1(x) + \left(\int_1^x \frac{y_1(\xi)}{W(\xi)} f(\xi)\,d\xi\right) y_2(x)$$

$$= \int_1^x \frac{-y_1(x)y_2(\xi) + y_1(\xi)y_2(x)}{W(\xi)} f(\xi)\,d\xi$$

$$= \int_1^x \frac{-\frac{1}{x}\xi^3 + \frac{1}{\xi}x^3}{4\xi}\left(\frac{1}{\xi} - \frac{3}{\xi^2}\right)d\xi$$

$$= \int_1^x \frac{x^4 - \xi^4}{4x\xi^2}\left(\frac{1}{\xi} - \frac{3}{\xi^2}\right)d\xi$$

Note: $G(x,\xi) = \begin{cases} \frac{x^4 - \xi^4}{4x\xi^2} & x < \xi \\ 0 & x > \xi \end{cases}$

$$y_p(x) = \int_1^x \frac{(x^4 - \xi^4)(\xi - 3)}{4x\xi^4}\,d\xi$$

$$= \int_1^x \frac{x^4\xi - 3x^4 - \xi^5 + 3\xi^4}{4x\xi^4}\,d\xi$$

$$= \int_1^x \left(\frac{x^3}{4\xi^3} - \frac{3x^3}{4\xi^4} - \frac{\xi}{4x} + \frac{3}{4x}\right)d\xi$$

$$= \left[-\frac{x^3}{8\xi^2} + \frac{x^3}{4\xi^3} - \frac{\xi^2}{8x} + \frac{3\xi}{4x}\right]_1^x$$

$$= \left(-\frac{1}{8}x + \frac{1}{4} - \frac{1}{8}x + \frac{3}{4}\right) - \left(-\frac{1}{8}x^3 + \frac{1}{4}x^3 - \frac{1}{8x} + \frac{3}{4x}\right)$$

$$= \left(\frac{1}{8} - \frac{1}{4}\right)x^3 + \left(-\frac{1}{8} - \frac{1}{8}\right)x + \left(\frac{1}{4} + \frac{3}{4}\right) + \left(\frac{1}{8} - \frac{3}{4}\right)\frac{1}{x}$$

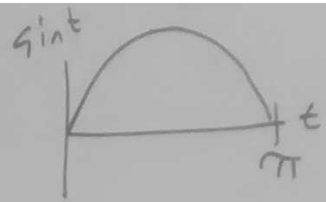
$$= \frac{1}{8}x^3 - \frac{1}{4}x + 1 - \frac{5}{8}\frac{1}{x}$$

$$y_g(x) = C_1 x^3 - \frac{1}{4}x + 1 - C_2\frac{1}{x}$$

BCs: Solve for constants

$$y(x) = -\frac{1}{60}x^3 - \frac{1}{4}x + 1 - \frac{11}{15}\frac{1}{x}$$

F014 #5



$$\int_0^{\pi} e^{x\sin t}\,dt$$

for $x \to \infty$: max contribution @ $t = \frac{\pi}{2}$. Expand @ $\frac{\pi}{2}$.

$$\int_0^{\pi} e^{x\left(1-\frac{1}{2}\left(t-\frac{\pi}{2}\right)^2\right)}\,dt$$

$$= e^{x}\int_0^{\pi} e^{-\frac{x}{2}\left(t-\frac{\pi}{2}\right)^2}\,dt$$

$$= \boxed{e^{x}\sqrt{\frac{2\pi}{x}}}$$

for $x \to -\infty$: max contribution at $t = 0, \pi$.

By Symmetry: $\int_0^{\pi} e^{x\sin t}\,dt = 2\int_0^{\frac{\pi}{2}} e^{x\sin t}\,dt$

Expand @ 0.

$$2\int_0^{\frac{\pi}{2}} e^{x\left(t-\frac{1}{6}t^3\right)}\,dt$$

$$\approx 2\int_0^{\frac{\pi}{2}} e^{xt}\,dt$$

$$= \frac{-2}{x}e^{xt}\Big|_{t=0}$$

$$= \boxed{-\frac{2}{x}}$$

F'014 #6 (Part 1)

$k_- \swarrow k \nearrow k_+$

$\lim_{x \to -\infty} |h_1(x)|^2 = 1$

$\lim_{x \to -\infty} |h_2(x)|^2 = 1$

$$\frac{d}{dx}\left[\frac{1}{k^2(\varepsilon x)} \frac{dh_1}{dx}\right] + h_1 = 0$$

$X := \varepsilon x \quad H_1(X) := h_1(x)$

$$\varepsilon \frac{d}{dX}\left[\frac{1}{k^2(X)} \varepsilon H_1'\right] + H_1 = 0$$

$$\varepsilon^2 \frac{1}{k^2} H_1'' - 2\varepsilon^2 \frac{k'}{k^3} H_1' + H_1 = 0$$

$$H_1 := \exp\left(\frac{1}{\varepsilon} u_0 + u_1 + \cdots\right)$$

$$\frac{\varepsilon^2}{k^2}\left[\left(\frac{1}{\varepsilon} u_0'' + u_1'' + \cdots\right) + \left(\frac{1}{\varepsilon} u_0' + u_1' + \cdots\right)^2\right]$$

$$- \frac{2\varepsilon^2 k'}{k^3}\left(\frac{1}{\varepsilon} u_0' + u_1' + \cdots\right) + 1 = 0$$

$O(1)$: $\frac{1}{k^2}(u_0')^2 + 1 = 0$

$$(u_0')^2 = -k^2$$

$$u_0' = \pm i k$$

$$u_0 = A \pm i \int^X k(t)\,dt$$

$O(\varepsilon)$: $\frac{u_0''}{k^2} + \frac{2}{k^2} u_0' u_1' - \frac{2k'}{k^3} u_0' = 0$

$$\frac{\pm i k'}{k^2} \pm \frac{2ik}{k^2} u_1' \mp \frac{2k'}{k^3} ik = 0$$

$$\frac{k'}{k^2} + 2k u_1' - 2k' = 0$$

$$2k u_1' = \left(2 - \frac{1}{k^2}\right) k'$$

$$u_1' = \left(\frac{1}{k} - \frac{1}{2k^3}\right) k'$$

$$u_1 = B + \ln k + \frac{1}{4k^2}$$

$$H_1 \approx \exp\left(\frac{A}{\varepsilon} \pm \frac{i}{\varepsilon} \int^X k(t)\,dt + B + \ln k + \frac{1}{4k^2}\right)$$

$$= k(X) \exp\left(\frac{A}{\varepsilon} + \frac{1}{4k^2}\right) \exp\left(\pm \frac{i}{\varepsilon} \int^X k(t)\,dt\right)$$

$$h_1(x) \approx k(\varepsilon x) \exp\left(\frac{A}{\varepsilon} + \frac{1}{4k^2(\varepsilon x)}\right) \exp\left(\pm \frac{i}{\varepsilon} \int^{\varepsilon x} k(t)\,dt\right)$$

$$1 = \lim_{x \to -\infty} |h_1(x)|^2 = k_-^2 \exp\left(\frac{2A}{\varepsilon} + \frac{1}{2k_-^2}\right) = k_-^2 \exp\left(\frac{1}{2k_-^2}\right) \exp\left(\frac{2A}{\varepsilon}\right) \Rightarrow \exp\left(\frac{2A}{\varepsilon}\right) = \frac{1}{k_-^2 \exp\left(\frac{1}{2k_-^2}\right)}$$

$$\lim_{x \to \infty} |h_1(x)|^2 = k_+^2 \exp\left(\frac{2A}{\varepsilon} + \frac{1}{2k_+^2}\right) = \frac{k_+^2 \exp\left(\frac{1}{2k_+^2}\right)}{k_-^2 \exp\left(\frac{1}{2k_-^2}\right)} = \boxed{\left(\frac{k_+}{k_-}\right)^2 \exp\left(\frac{1}{2}\left[\frac{1}{k_+^2} - \frac{1}{k_-^2}\right]\right)}$$

F'014 #6 (Part 2)

$k_- \swarrow k \nearrow k_+$

$\lim_{x\to-\infty} |h_1(x)|^2 = 1$

$\lim_{x\to\infty} |h_2(x)|^2 = 1$

$$\frac{1}{k^2(\varepsilon x)} h_2'' + h_2 = 0$$

$$X := \varepsilon x \qquad H_2(X) := h_2(x)$$

$$\frac{\varepsilon^2}{k^2} H_2'' + H_2 = 0$$

$$H_2 := \exp\left(\frac{1}{\varepsilon} u_0 + u_1\right)$$

$$\frac{\varepsilon^2}{k^2}\left[\left(\tfrac{1}{\varepsilon}u_0'' + u_1''\right) + \left(\tfrac{1}{\varepsilon}u_0' + u_1'\right)^2\right] + 1 = 0$$

$O(1)$: $$\frac{(u_0')^2}{k^2} + 1 = 0$$

$$(u_0')^2 = -k^2$$

$$u_0' = \pm ik$$

$$u_0 = A \pm i \int^X k(t)dt$$

$O(\varepsilon)$: $$\frac{u_0''}{k^2} + \frac{2}{k^2} u_0' u_1' = 0$$

$$\pm\frac{ik'}{k^2} \pm \frac{2ik}{k^2} u_1' = 0$$

$$k' + 2ku_1' = 0$$

$$u_1' = -\frac{k'}{2k}$$

$$u_1 = B - \tfrac{1}{2}\ln k$$

$$H_2 = \exp\left(\frac{A}{\varepsilon} \pm \frac{i}{\varepsilon}\int^X k(t)dt + B - \tfrac{1}{2}\ln k\right)$$

$$= \frac{1}{\sqrt{k}} \exp\left(\frac{A}{\varepsilon} + B\right)\exp\left(\pm\frac{i}{\varepsilon}\int^X k(t)dt\right)$$

$$h_2(x) = \frac{1}{\sqrt{k(\varepsilon x)}} \exp\left(\frac{A}{\varepsilon} + B\right)\exp\left(\pm\frac{i}{\varepsilon}\int^{\varepsilon x} k(t)dt\right)$$

$$1 = \lim_{x\to-\infty}|h_2(x)|^2 = \frac{1}{k_-}\exp\left(\frac{2A}{\varepsilon} + 2B\right) \Rightarrow \exp\left(\frac{2A}{\varepsilon} + 2B\right) = k_-$$

$$\lim_{x\to\infty}|h_2(x)|^2 = \frac{1}{k_+}\exp\left(\frac{2A}{\varepsilon} + 2B\right) = \boxed{\frac{k_-}{k_+}}$$