UC Davis
Applied Math
Prelim Solutions
"Back of the Napkin" style

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If you find a mistake, please send the correct solution to the author.

F'013 #1

$$h'' + k^2 h = 0$$

$$k = k(x) = k_1 + (k_2 - k_1)\tanh\left(\frac{x}{L}\right) \qquad 0<k_1<k_2$$

$$X := \frac{x}{L} \qquad \kappa(X) := k(x) \qquad H(X) := h(x)$$

$$\frac{1}{L^2}H'' + \kappa^2 H = 0$$

$$H := \exp(L u_0 + u_1)$$

$$\frac{1}{L^2}\left[(L u_0'' + u_1'') + (L u_0' + u_1')^2\right] + \kappa^2 = 0$$

Assume κ is $O(1)$

$O(1)$: (eikonal)

$$(u_0')^2 + \kappa^2 = 0$$

$$u_0' = \pm i\kappa$$

$$u_0 = C \pm i\int^X \kappa(t)\,dt$$

$O(\varepsilon)$: (transport)

$$u_0'' + 2u_0' u_1' = 0$$

$$\pm i\kappa' \pm 2i\kappa u_1' = 0$$

$$\kappa' + 2\kappa u_1' = 0$$

$$u_1' = -\frac{1}{2}\frac{\kappa'}{\kappa}$$

$$u_1 = D - \frac{1}{2}\ln\kappa$$

$$H = \exp\left(CL \pm iL\int^X \kappa(t)\,dt + D - \frac{1}{2}\ln\kappa\right)$$

$$= \frac{1}{\sqrt{\kappa}}\exp\left(CL + D \pm iL\int^X \kappa(T)\,dT\right.$$

$$h(x) = \frac{1}{\sqrt{k(x)}}\exp\left(CL + D \pm i\int^x k(t)\,dt\right)$$

$$= \frac{1}{\sqrt{k(x)}}\exp(CL+D)\exp\left(\pm i\int^x k(t)\,dt\right)$$

In part (c), we are given info to conclude that $(k_2 - k_1) \approx 0$ } odd. error in question?

$$\therefore\ k(t) \approx k_1$$

$$A^2 = \lim_{x\to-\infty}|h(x)|^2 = \frac{1}{k_1}\exp(2CL + 2D)$$

$$\therefore\ h(x) = A\exp(ik_1 x)$$

F'013 #2

$$\varepsilon y'' + \sqrt{x}\, y' + y = 0$$

$$y(0) = 0, \; y(1) = 1$$

INNER SOLUTION

Let $x := \varepsilon^{\alpha} X$, $Y(X) := y(x)$

$$\varepsilon^{1-2\alpha} Y'' + \varepsilon^{-\frac{1}{2}\alpha} \sqrt{X}\, Y' + Y = 0, \; Y(0) = 0$$

$$\alpha := \frac{2}{3}$$

$1 - 2\alpha = -\frac{1}{2}\alpha$

$1 = \frac{3}{2}\alpha$

$\frac{2}{3} = \alpha$

$$\varepsilon^{-\frac{1}{3}} Y'' + \varepsilon^{-\frac{1}{3}} \sqrt{X}\, Y' + Y = 0$$

$$Y'' + \sqrt{X}\, Y' + \varepsilon^{\frac{1}{3}} Y = 0$$

$O(1)$: $Y'' + \sqrt{X}\, Y' = 0$

$$\frac{Y''}{Y'} = -\sqrt{X}$$

$$\ln Y' = -\frac{2}{3} X^{3/2} + c$$

$$Y' = c e^{-\frac{2}{3} X^{3/2}}$$

$$Y = c \int_0^X e^{-\frac{2}{3} X^{\frac{3}{2}}} dX + d$$

$Y(0) = 0 \Rightarrow d = 0$

$$\therefore Y = c \int_0^X e^{-\frac{2}{3} t^{\frac{3}{2}}} dt$$

OUTER SOLUTION

$\sqrt{x}\, y' + y = 0 \qquad y(1) = 1$

$$\frac{y'}{y} = -\frac{1}{\sqrt{x}}$$

$$\ln y = -2\sqrt{x} + a$$

$$y = a e^{-2\sqrt{x}}$$

$$1 = y(1) = a e^{-2} \Rightarrow a = e^{2}$$

$$\therefore y(x) = e^{2 - 2\sqrt{x}}$$

MATCH

$$\lim_{X \to \infty} Y(X) = \lim_{x \to 0^+} y(x)$$

$$c \int_0^\infty e^{-\frac{2}{3} t^{\frac{3}{2}}} dt = e^{2}$$

$$c = \frac{e^2}{\int_0^\infty e^{-\frac{2}{3} t^{3/2}} dt}$$

SOLUTION

$$y(x) + Y\left(\frac{x}{\varepsilon^{2/3}}\right) - e^2$$

$$\boxed{e^{2 - 2\sqrt{x}} + \frac{e^2}{\int_0^\infty e^{-\frac{2}{3} t^{3/2}} dt} \int_0^{\frac{x}{\varepsilon^{2/3}}} e^{-\frac{2}{3} t^{\frac{3}{2}}} dt - e^2}$$

solution contributed by Guga Mikaberidze (Mar 2020)

3) $u(r,t) = R(r)\,T(t)$

a)

$$RT'' = a^2 R''T + \frac{a^2}{r}R'T$$

$$\frac{T''}{T} = a^2\frac{R''}{R} + \frac{a^2}{r}\frac{R'}{R}$$

$$\frac{T''}{T} = -\lambda^2$$ (because soln's must oscillate)

$$a^2\frac{R''}{R} + \frac{a^2}{r}\frac{R'}{R} = -\lambda^2$$

$$R'' + \frac{1}{r}R' + \frac{\lambda^2}{a^2}R = 0$$

$$\frac{1}{p}(pR')' = \frac{pR''}{p} + \frac{p'R'}{p} \qquad p = r$$

$$\boxed{\frac{1}{r}(rR')' + \left(\frac{\lambda}{a}\right)^2 R = 0}$$

b)

$$\frac{1}{r}(rR')' + \left(\frac{\lambda}{a}\right)^2 R = 0$$

$$\frac{1}{x}(xJ_n')' + \left(1 - \frac{n^2}{x^2}\right)J_n = 0 \qquad \text{Bessel}$$

$J_0 : \quad n = 0$

$$\frac{1}{x}(xJ_0')' + J_0 = 0$$

$$r = x\frac{a}{\lambda} \qquad R(r) = J_0\left(\frac{\lambda r}{a}\right)$$

$$R(r_0) = 0 = J_0\left(\frac{\lambda r_0}{a}\right)$$

Zeros of Bessel $J_0(x_i) = 0$

$$\frac{\lambda_i r_0}{a} = x_i \qquad \boxed{\lambda_i = \frac{a x_i}{r_0}}$$

$$R_i(r) = J_0\left(\frac{\lambda_i r}{a}\right) = \boxed{J_0\left(\frac{x_i r}{r_0}\right) = R_i(r)}$$

F'013 #4 (Part 1)

$$f'' + \lambda f = 0$$

$$f'(0) = f(\ell) = 0$$

Claim: $\lambda > 0$

pf/
$$0 < \frac{\int_0^\ell (f')^2 dx}{\int_0^\ell f^2 dx} = \frac{-\int_0^\ell f f'' dx + [ff']_0^\ell}{\int_0^\ell f^2 dx}$$

$$= \frac{-\int_0^\ell f(-\lambda f) dx}{\int_0^\ell f^2 dx} = \frac{\lambda \int_0^\ell f^2 dx}{\int_0^\ell f^2 dx} = \lambda \quad //$$

wlog let $\lambda = \frac{\pi^2 n^2}{4\ell^2}$, $n \in \mathbb{R}$.

$$f'' + \frac{\pi^2 n^2}{4\ell^2} f = 0$$

solved by $f(x) = a\cos\frac{\pi n}{2\ell}x + b\sin\frac{\pi n}{2\ell}x$

$$0 = f(\ell) = a\cos\frac{\pi}{2}n + b\sin\frac{\pi}{2}n$$

$$= a\cos\frac{\pi}{2}n \Rightarrow a = 0 \text{ or } n \in 2\mathbb{Z}-1$$

$$0 = f'(0) = -\frac{\pi n a}{2\ell}\sin\frac{\pi n}{2\ell}0 + \frac{\pi n b}{2\ell}\cos\frac{\pi n}{2\ell}0$$

$$= \frac{\pi n b}{2\ell} \Rightarrow b = 0.$$

$f(x) = 0$ or $f(x) = a\cos\frac{\pi(2k-1)}{2\ell}x$, $k \in \mathbb{N}$

Normalize: $a^2\int_0^\ell \cos^2\frac{\pi(2k-1)}{2\ell}x\,dx = \frac{\ell}{2}a^2 := 1 \Rightarrow a = \sqrt{\frac{2}{\ell}}$

$$\boxed{\begin{array}{l} \text{eigenvalues: } \lambda_k = \frac{\pi^2(2k-1)^2}{4\ell^2}, k \in \mathbb{N} \\ \text{eigenfunctions: } \sqrt{\frac{2}{\ell}}\cos\frac{\pi(2k-1)}{2\ell}x \end{array}}$$

F'013 #4 (Part 2)

approximate $\ell^2 - x^2$ with

$$\{\varphi_k\} = \left\{\sqrt{\frac{2}{\ell}} \cos \frac{\pi(2k-1)}{2\ell} x\right\}$$

project: $g \mapsto \sum_{k=1}^{\infty} \langle \varphi_k, g \rangle \varphi_k$. Let $\kappa := \frac{\pi(2k-1)}{2\ell}$

$$\langle \varphi_k, g \rangle = \sqrt{\sqrt{\frac{2}{\ell}} \int_0^{\ell} (\ell^2 - x^2) \cos \kappa x \, dx}$$

$$= \sqrt[4]{\frac{2}{\ell}} \sqrt{\left[-\frac{x^2}{\kappa} \sin \kappa x - \frac{2x}{\kappa^2} \cos \kappa x + \frac{2}{\kappa^3} \sin \kappa x\right]_0^{\ell}}$$

$$= \sqrt[4]{\frac{2}{\ell}} \sqrt{\frac{-x^2}{\kappa}(-1)^{k+1} + \frac{2}{\kappa^3}(-1)^{k+1} + \frac{2x}{\kappa^2}}$$

(put these pieces together for answer)

F '013 # 5 Page 1

$\dot{x} = ax + y - x f(x^2+y^2)$ $\qquad$ $a \in \mathbb{R}$, $f \in C^0$, $f(0)=0$, $f(u) \geq u^{1/2}$

$\dot{y} = -x + ay - y f(x^2+y^2)$

$$2x\dot{x} + 2y\dot{y} = 2ax^2 + 2xy - x^2 f(x^2+y^2) - 2xy + 2ay^2 - y^2 f(x^2+y^2)$$
$$= 2a(x^2+y^2) - (x^2+y^2) f(x^2+y^2) = (x^2+y^2)[2a - f(x^2+y^2)]$$

fixed points:

$$2x(0) + 2y(0) = (x^2+y^2)[2a - f(x^2+y^2)]$$

$x^2+y^2 = 0$ or $f(x^2+y^2) = 2a$

$(0,0)$

would imply $\begin{cases} 0 = \dot{x} = ax + y - 2ax = y - ax \\ 0 = \dot{y} = -x + ay - 2ay = -x - ay \end{cases}$

$\therefore$ $y = ax$ and $x = -ay$

$\therefore$ $y = -a^2 y$

$\therefore$ $y = 0$, $x = 0$

$\therefore$ only fixed pt: $(0,0)$

Linear stability: $J(x,y) = \begin{bmatrix} a - f(x^2+y^2) - 2x^2 f'(x^2+y^2) & 1 - 2xy f(x^2+y^2) \\ -1 - 2xy f(x^2+y^2) & a - f(x^2+y^2) - 2y^2 f'(x^2+y^2) \end{bmatrix}$

assume $f \in C^1$

$$J(0,0) = \begin{bmatrix} a & 1 \\ -1 & a \end{bmatrix}$$

$$\lambda = a \pm i$$

- stable spiral for $a < 0$

unstable spiral for $a > 0$

F '013 #5 Page 2

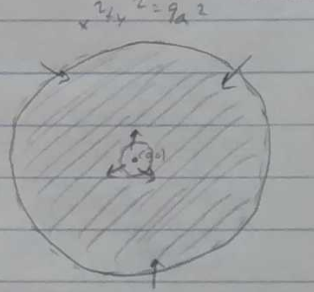
We have that $(0,0)$ is the only fixed pt and it is a spiral source for $a>0$.

Now recall $2x\dot{x}+2y\dot{y}=(x^2+y^2)[2a-f(x^2+y^2)]$

$$\Rightarrow \frac{d}{dt}[x^2+y^2]=(x^2+y^2)[2a-f(x^2+y^2)]$$

Consider the closed region bdd by $x^2+y^2=9a^2$ with the neighborhood of $(0,0)$ removed.

On the outer bdy of this region,



$$\frac{d}{dt}[x^2+y^2]=(x^2+y^2)[2a-f(x^2+y^2)]$$

$$=9a^2[2a-f(9a^2)]$$

$$\leq 9a^2[2a-3a]$$

$$=-9a^3$$

$$<0.$$ $\therefore$ all trajectories flow inward

On the inner bdy of this region, all trajectories flow inward (spiral source at $(0,0)$)

There are no fixed pts inside the region.

$\therefore$ By P-B a limit cycle must exist.

Special case: $f(u)=u^{1/2}$.

$$0=\frac{d}{dt}[x^2+y^2]=(x^2+y^2)[2a-f(x^2+y^2)]$$

$0=x^2+y^2$ or $0=2a-f(x^2+y^2)$

not in region

$$2a=f(x^2+y^2)$$

$$2a=\sqrt{x^2+y^2}$$

$$\boxed{4a^2=x^2+y^2}$$

F '013 #6

$$\frac{d^2r}{d\theta^2} + r = \frac{1}{L} + \varepsilon L r^2$$

$$\begin{cases} \dfrac{dr}{d\theta} := s \\ \dfrac{ds}{d\theta} = -r + \dfrac{1}{L} + \varepsilon L r^2 \end{cases}$$

fixed pts: $\begin{cases} 0 = s \\ 0 = -r + \frac{1}{L} + \varepsilon L r^2 \end{cases}$

$$\varepsilon L r^2 - r + \frac{1}{L} = 0$$

$$r = \frac{1 \pm \sqrt{1-4\varepsilon}}{2\varepsilon L}$$

$$J(r,s) = \begin{bmatrix} 0 & 1 \\ -1+2\varepsilon L r & 0 \end{bmatrix}$$

$$J\left(\frac{1 \pm \sqrt{1-4\varepsilon}}{2\varepsilon L}, 0\right) = \begin{bmatrix} 0 & 1 \\ \pm\sqrt{1-4\varepsilon} & 0 \end{bmatrix}$$

saddle at $\left(\frac{1+\sqrt{1-4\varepsilon}}{2\varepsilon L}, 0\right)$ as $\varepsilon \to 0^+$: $(\infty, 0)$

center at $\left(\frac{1-\sqrt{1-4\varepsilon}}{2\varepsilon L}, 0\right)$ as $\varepsilon \to 0^+$: $\left(\frac{1}{L}, 0\right)$
(conservative system)