PRELIMINARY EXAM IN ANALYSIS, SPRING 2019

All problems are worth the same amount. You should give full proofs or explanations of your solutions. Remember to state or cite theorems that you use in your solutions.

Important: Please use a different sheet for the solution to each problem.

1. Consider the Hilbert space $L^2([0,2\pi])$ of complex-valued square integrable functions with the inner product given by

$$(f,g) = \int_0^{2\pi} \overline{f(x)} g(x) dx.$$

(a) For all $\phi \in \mathbb{R}$, define $g_\phi \in L^2([0,2\pi])$, by $g_\phi(\theta) = \sin(\theta - \phi)$, for $\theta \in [0,2\pi]$. Let V be the closed linear span of $\{g_\phi \mid \phi \in \mathbb{R}\}$. Show that V is two-dimensional.

(b) Find $k : [0,2\pi] \times [0,2\pi] \to \mathbb{C}$ such that for all $f \in L^2([0,2\pi])$, the integral operator K defined by

$$Kf(x) = \int_0^{2\pi} k(x,y) f(y) dy,$$

satisfies

$$\|Kf - f\| = \inf\{\|g - f\| \mid g \in V\}.$$

2. Let $S = [0,1] \times [0,1]$ and consider the space $C(S)$ of continuous complex-valued functions on S, equipped with the supremum norm. Define $F \subset C(S)$ by

$$F = \{f \mid \exists n \geq 1, g_1, \ldots, g_n, h_1, \ldots, h_n \in C([0,1]) \text{ such that } f(x,y) = \sum_{k=1}^{n} g_k(x) h_k(y)\}.$$

Show that F is dense in C(S).

3. Choose $f = a(2\chi_{[-\frac{\pi}{2},\frac{\pi}{2}]} - 1) \in L^2\mathbb{T} = L^2(-\pi,\pi]$ with a real and $f_n = f * f * \ldots * f$ the n-fold convolution so that $\{f_n\}$ converges in $\|\cdot\|_{L^2}$ to a nonzero function g. Find g.

4. Show that if K is a compact self adjoint linear operator on a separable Hilbert space with closed image then the image is finite dimensional.

5. Consider a C^1 function $f : \mathbb{R}^2 \to \mathbb{R}$. Suppose $f \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ and $|\nabla f| \in L^2(\mathbb{R}^2)$. Show that there exists a constant $K < 10$ such that the following inequality holds:

$$\|f\|_{L^2(\mathbb{R}^2)}^2 \leq K \|f\|_{L^1(\mathbb{R}^2)} \|\nabla f\|_{L^2(\mathbb{R}^2)}.$$

6. Let $\Omega = (0,1) \subset \mathbb{R}$. For $\bar{u} = \int_\Omega u(x)\, dx$, show that

$$\|u - \bar{u}\|_{L^\infty(\Omega)} \leq \|u'\|_{L^2(\Omega)}, \qquad \forall u \in W^{1,1}(\Omega).$$

(Hint: The average $\bar{u} = u(x_0)$ for some $x \in [0,1]$.)

PRELIMINARY EXAM IN ANALYSIS
SPRING 2018

All problems are worth the same amount. You should give full proofs or explanations of your solutions. Remember to state or cite theorems that you use in your solutions.

Important: Please use a different sheet for the solution to each problem.

1. Let $L^2([0,1])$ be the Hilbert space of complex-valued square integrable functions with inner product

$$\langle f, g\rangle = \int_0^1 \overline{f(x)}g(x)dx.$$

Does $L^2([0,1])$ have an orthonormal basis $\{u_n | n \geq 0\}$ such that each $u_n(x)$ is a polynomial of degree $2n$ using only even powers of x, i.e.,

$$u_n(x) = a_0 + a_1x^2 + a_2x^4 + \cdot + a_nx^{2n}?$$

Construct such a basis if it exists, or prove that it does not exist.

2. Let $\mathcal{H}$ be a Hilbert space and let $P, Q \in \mathcal{B}(\mathcal{H})$ be two orthogonal projections. Prove that $\ker PQ \subseteq \ker P + \ker Q$ always, and that $\ker PQ = \ker P + \ker Q$ when PQ is also an orthogonal projection.

3. Let X be a Banach space. Suppose that $f : (0,1) \to \mathcal{B}(X)$ is a differentiable function in the sense that the limit

$$f'(t) = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}$$

exists in the norm topology on $\mathcal{B}(X)$ for each t. Prove that $e^{f(t)}$ is also differentiable in the same sense, and that its derivative satisfies the inequality

$$\left\| \frac{d}{dt} e^{f(t)} \right\| \leq \|f'(t)\| e^{\|f(t)\|}.$$

4. Let T be a bounded linear operator on a Hilbert space with an orthonormal basis of eigenvectors with eigenvalues $\Lambda = \{\lambda_n\}$. Show that the spectrum $\sigma(T)$ is exactly the closure of the set Λ.

5. Let a be an irrational real number and let $f \in L^2(\mathbb{T}) = L^2(\mathbb{R}/2\pi\mathbb{Z})$ be a square-integrable function on the circle such that $f(\theta) = f(\theta + \pi a)$ as elements of $L^2(\mathbb{T})$. Prove that f is essentially constant.

6. Given $t > 0$ and given $\vec{x} \in \mathbb{R}^3$, let

$$K_t(\vec{x}) = \frac{e^{-|\vec{x}|^2/t}}{t}.$$

Show that if $f \in L^3(\mathbb{R}^3)$, then the convolution $K_t * f$ lies in $L^\infty(\mathbb{R}^3)$, and that its norm is bounded by a constant independent of t.

PRELIMINARY EXAM IN ANALYSIS
FALL 2018

All problems are worth the same amount. You should give full proofs or explanations of your solutions. Remember to state or cite theorems that you use in your solutions.

Important: Please use a different sheet for the solution to each problem.

1. Let $f : [0,1] \to \mathbb{R}$ be a continuous function. Prove that

$$\lim_{n\to\infty} \int_0^1 f(x)\sin(n\pi x)dx = 0.$$

2. Consider the function $f : [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x\log x & \text{if } x \in (0,1] \\ 0 & \text{if } x = 0. \end{cases}$$

Be sure to justify your answer for each of the following questions.

(a) Is f Lipschitz continuous on $[0,1]$?

(b) Is f uniformly continuous on $[0,1]$?

(c) Suppose (p_n) is a sequence of polynomial functions on $[0,1]$, converging uniformly to f. Is the set $A = \{p_n \mid n \geq 1\} \cup \{f\}$ equicontinuous?

3. Show that for every $f \in C(\mathbb{T})$ and $\varepsilon > 0$ there is an initial condition $g \in C(\mathbb{T})$ for which there is a solution $u(x,t)$ to the heat equation on a ring with $u(x,0) = g(x)$ and $|u(x,1) - f(x)| < \varepsilon$ for every $x \in \mathbb{T}$.

4. Consider the functions $f_N(x) = (2\pi)^{-1}\sum_{n=-N}^{N} e^{inx}$. Show that if $g \in L^2(\mathbb{T})$ then $\{f_N * g\}$ converges in $\|\cdot\|_{L^2}$-norm to g (here, $*$ denotes convolution).

5. Show the following: For $u \in L^1(\mathbb{R}^n)$ there holds

$$\lim_{h\to 0} \|u(x+h) - u(x)\|_{L^1(\mathbb{R}^n)} = 0.$$

6. Let $\Omega = \{(x,y) : y \geq 0, x \in \mathbb{R}\}$. Let $f = C_c^1\left(\mathbb{R}^2\right)$ (space of continuous functions with compact support and with continuous first derivatives). Show the following

$$\int_{\mathbb{R}} |f(x,0)|^2 dx \leq 2\left(\int_{\Omega} |f(x,y)|^2 dxdy + \int_{\Omega} \left|\frac{\partial f}{\partial y}(x,y)\right|^2 dxdy\right).$$

PRELIMINARY EXAM IN ANALYSIS
Spring, 2017

Instructions:

(1) All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.
(2) Use separate sheets for the solution of each problem.

Problem 1. Define $f_n : [0, \infty) \to \mathbb{R}$ by

$$f_n(x) = \sin \sqrt{x + 4n^2\pi^2},$$

and let $\mathcal{F} = \{f_n : n \in \mathbb{N}\} \subset C([0, \infty))$.
(a) Show that $\mathcal{F}$ is equicontinuous in $C([0, \infty))$.
(b) Show that the sequence (f_n) converges pointwise but not uniformly on $[0, \infty)$.
(c) Is $\mathcal{F}$ totally bounded in the space $C_b([0, \infty))$ of bounded, continuous functions $f : [0, \infty) \to \mathbb{R}$, equipped with the sup-norm?

Problem 2. Suppose that X is a metric space with metric d such that every continuous function $f : X \to \mathbb{R}$ is bounded. Prove that X is complete.

Problem 3. Let A be a linear operator on a Banach space B that maps any strongly converging sequence into a weakly converging one. Prove that A is a bounded operator.

Problem 4. Let $K(x, y)$ be a continuous function on the unit square. Prove that A is a compact operator acting on $L^2[0, 1]$, where

$$(Af)(x) = \int_0^1 K(x, y)f(y)dy.$$

Problem 5. Prove the Riemann-Lebesgue Lemma, namely if $f \in L^1(\mathbb{R}^n)$ then $\widehat{f}(\xi) \to 0$ as $|\xi| \to \infty$, where

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i\xi\cdot x} f(x)dx.$$

Problem 6. Give an example of a sequence of functions that converges weakly in L^2, strongly in L^1, but does not converge strongly in L^2. Be sure to justify all of your assertions.

PRELIMINARY EXAM IN ANALYSIS
FALL 2017

All problems are worth the same amount. You should give full proofs or explanations of your solutions. Remember to state or cite theorems that you use in your solutions.

Important: Please use a different sheet for the solution to each problem.

1. Prove that every metric subspace of a separable metric space is separable.

2. Let X be a Banach space with dual space X^* and let $A \subseteq X$ be a linear subspace. Define the annihilator $A^\perp \subseteq X^*$ of A by

$$A^\perp = \{f \in X^* : f(x) = 0 \text{ for all } x \in A\}.$$

Prove that A is dense in X if and only if $A^\perp = \{0\}$.

3. Prove or disprove the following statement: If $f \in C^\infty([0,1])$ is a smooth function, then there is a sequence (p_n) of polynomials on $[0,1]$ such that $p_n^{(k)} \to f^{(k)}$ uniformly on $[0,1]$ as $n \to \infty$ for every integer $k \geq 0$. Here $f^{(k)}$ denotes the kth derivative of f.

4. Let $[a,b] \subseteq \mathbb{R}$ be a closed interval and let

$$\|f\|_\infty = \sup_{x \in [a,b]} |f(x)| \qquad \|f\|_2 = \sqrt{\int_a^b |f(x)|^2 dx}$$

denote the L^∞ and L^2 norms. If $f \in C^1([a,b])$, prove that

$$\|f\|_\infty^2 \leq \frac{\|f\|_2^2}{b-a} + 2\|f\|_2 \|f'\|_2.$$

5. Let A be a bounded self-adjoint operator on a Hilbert space $\mathcal{H}$. Prove that

$$\exp(iA) = \sum_{n=0}^{\infty} \frac{1}{n!}(iA)^n$$

is also a well-defined bounded operator on $\mathcal{H}$ and is unitary.

6. Consider a linear functional $\phi(f) = f(1/2)$ defined on the space of polynomials on $[0,1]$. Does ϕ extend to a bounded linear functional on $L^2([0,1])$? Prove or disprove.

PRELIMINARY EXAM IN ANALYSIS
Spring, 2016

Instructions:

1. All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.

2. Use separate sheets for the solution of each problem.

Problem 1. Let $f(x)$ be a continuous function on $\mathbb{R}$ such that for any polynomial $P(x)$ we have

$$\int_{\mathbb{R}} f(x)P(x)dx = 0.$$

Show that $f(x)$ is identically zero.

Problem 2. Let M be a multiplication on $L^2(\mathbb{R})$ defined by

$$Mf(x) = m(x)f(x),$$

where $m(x)$ is continuous and bounded. Prove that M is a bounded operator on $L^2(\mathbb{R})$ and that its spectrum is given by

$$\sigma(M) = \{m(x) : x \in \mathbb{R}\}^{cl},$$

where A^{cl} denotes the closure of A. Can M have eigenvalues?

Problem 3. Show that the closed unit ball of a Hilbert space H is compact if and only if dim H is finite.

Problem 4. Suppose f is a function in the Schwartz space $\mathcal{S}(\mathbb{R})$ which satisfies the normalizing condition $\int\limits_{-\infty}^{+\infty} |f(x)|^2 dx = 1$. Let $\hat{f}$ denote the Fourier transform of f. Show that

$$\left(\int_{-\infty}^{+\infty} x^2|f(x)|^2dx\right)\left(\int_{-\infty}^{+\infty} \omega^2|\hat{f}(\omega)|^2d\omega\right) \geq \frac{1}{16\pi^2}.$$

Problem 5. Let $f(x) \in W^{1,1}([0,1])$. Let $\bar{f} = \int_0^1 f(x)dx$. Show that

$$\|f - \bar{f}\|_{L^1([0,1])} \leq 2\|f(x)x(1-x)\|_{L^1([0,1])}.$$

Problem 6. Let H be a Hilbert space and let U be a unitary operator, that is surjective and isometric, on H. Let $I = \{v \in H : Uv = v\}$ be the subspace of invariant vectors with respect to U.

a) Show that $\{Uw - w : w \in H\}$ is dense in $I^{\perp}$ and that I is closed.

b) Let P be the orthogonal projection onto I. Show that

$$\frac{1}{N}\sum_{n=1}^{N} U^n v \to Pv.$$

PRELIMINARY EXAM IN ANALYSIS
FALL, 2016

All problems are worth the same amount. You should give full proofs or explanations of your solutions. Remember to state or cite theorems that you use in your solutions.

Important: Please use a different sheet for the solution to each problem.

1. Let A and B be two bounded, self-adjoint operators on a Hilbert space $\mathcal{H}$. Prove that

$$||Af||\,||Bf|| \geq \frac{|\langle [A,B]f,f\rangle|}{2},$$

where $[A,B] = AB - BA$ is the commutator of A and B. In addition, prove that equality holds if and only if $Af = cBf$ for some $c \in \mathbb{R}$.

2. Let $f : \mathbb{T} \to \mathbb{C}$ be a C^1 function such that $\int\limits_{-\pi}^{\pi} f(x)dx = 0$. Show that

$$\int_{-\pi}^{\pi} |f(x)|^2 dx \leq \int_{-\pi}^{\pi} |f'(x)|^2 dx.$$

3. Let $y = \{a_n\}_{n=1}^{\infty}$ be a sequence of real-valued scalars and assume that the series $\sum_{n=1}^{\infty} a_n x_n$ converges for every $x \in \ell^2(\mathbb{N})$. Show that $y \in \ell^2(\mathbb{N})$.

4. Let $f \in L^1(\mathbb{R})$ and assume that $\hat{f}$, the Fourier transform of f, is supported on the interval $[-\frac{1}{2}, \frac{1}{2}]$. Let $\operatorname{sinc}(x) = \frac{\sin x}{x}$. Prove that

$$f(x) = \sum_{n\in\mathbb{Z}} f(n)\operatorname{sinc}(x-n),$$

where the equality holds in the L^2-sense. (Here, the Fourier transform of a function g is given by $\hat{g}(\omega) = \int\limits_{-\infty}^{\infty} g(x)e^{-2\pi i x\omega}dx$.)

5. Let $K : [0,1]\times[0,1] \to \mathbb{R}$ be a continuous function and fix $1 < p < \infty$. Given $f \in L^p([0,1])$, define $Tf : [0,1] \to \mathbb{R}$ by

$$Tf(x) = \int_0^1 K(x,y)f(y)dy.$$

(a) Prove that Tf is a continuous function.

(b) Prove that the image under T of the unit ball in $L^p([0,1])$ is precompact in $C([0,1])$.

6. Let D denote the closed unit disk in $\mathbb{C}$, and consider the complex Hilbert space

$$\mathcal{H} \stackrel{\text{def}}{=} \Big\{ f : D \to \mathbb{C} \,\Big|\, f(z) = \sum_{k=0}^{\infty} a_k z^k \text{ and } ||f||_{\mathcal{H}}^2 \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} (1+k^2)|a_k|^2 < \infty \Big\}.$$

Prove that the linear functional $L : \mathcal{H} \to \mathbb{C}$ defined by $L(f) = f(1)$ is bounded, and find an element $g \in \mathcal{H}$ such that $L(f) = \langle g, f\rangle$. (In other words, so that g represents L as in the Riesz representation theorem.)

Analysis Prelim Problems
Spring, 2015

Instructions:

1. *All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*
2. *Use separate sheets for the solution of each problem.*

Problem 1. Let $\mathcal{H}$ be a separable Hilbert space. A sequence (x_n) in $\mathcal{H}$ *converges in the Cesàro sense* to $x \in \mathcal{H}$ if the averages of its partial sums converge strongly to x, i.e., if

$$\bar{x}_N = \frac{1}{N}\sum_{n=1}^{N} x_n \to x \qquad \text{as } N \to \infty.$$

(a) Prove that if (x_n) converges strongly to x in $\mathcal{H}$, then (x_n) also converges in the Cesàro sense to x.

(b) Give an example of a sequence that converges in the Cesàro sense but does not converge weakly.

(c) Give an example of a sequence that converges weakly but does not converge in the Cesàro sense.

Problem 2.

Suppose $f : [-1, 1] \to \mathbb{R}$ is an odd continuous function such that $f(-1) = f(1)$. Given that $\int_{-1}^{1} \sin(nx) f(x)dx = 0$ for all positive integers n, show that $f \equiv 0$.

Problem 3.

Let $P(x) : \mathbb{R} \to \mathbb{R}$ be a polynomial of degree n. Show that there exists a constant C depending only on n such that $|P(\xi)| \leq C \int_{-1}^{1} |P(x)|^2 dx$ for all $\xi \in (-1, 1)$. (Remark: It may help to consider this problem from a functional analytic perspective.)

Problem 4.

Let $f_n : [0, 1] \to \mathbb{R}$ be a sequence of measurable functions. Suppose

(i) $\int_0^1 |f_n(x)|^2 dx \leq 1$ for $n = 1, 2, \cdots$.

(ii) $f_n \to 0$ almost everywhere.

Show that

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = 0.$$

Problem 5.

Find the Fourier series of the $2L$-periodic extension of

$$f(x) = \begin{cases} x & \text{if } x \in [0, L] \\ 0 & \text{if } x \in (-L, 0] \end{cases}$$

.

Show that

$$\pi^2/8 = \sum_{m=0}^{\infty} \frac{1}{(2m+1)^2}.$$

Problem 6.

Let $\{A_n\}$ be a sequence of bounded linear operators on a Hilbert space H that converges weakly to an operator A, and suppose that for each each $x \in H$ one has $\|A_n x\| \to \|Ax\|$ as $n \to \infty$. Prove that A_n strongly converges to A (in particular, for unitary operators weak convergence to a unitary operator implies strong convergence).

PRELIMINARY EXAM IN ANALYSIS

Fall, 2015

Instructions:

1. All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.

2. Use separate sheets for the solution of each problem.

Problem 1.

Consider the following sequences of functions parametrized by n:

- $a_n(x) = e^{i2\sqrt{n}\pi x}, \quad x \in [0, 1],$

- $b_n(x) = \sqrt{n}e^{-n|x|}, \quad x \in \mathbb{R},$

- $c_n(x) = ne^{-nx^2}, \quad x \in \mathbb{R},$

- $d_n(x) = \sum_{k=-n}^{n} e^{i2k\pi x}, \quad x \in [0, 1].$

As n tends to infinity, which sequences converge (a) almost everywhere, (b) L^2-strongly, (c) L^2-weakly but not strongly. Explain your answer.

Problem 2. Let T be a linear operator on a Banach space. Show that T is bounded *if and only if* T is continuous.

Problem 3. Let $f : [0, 1] \to \mathbb{R}$ be a C^1-function and suppose that $|f'(x)| \geq 1$ for all $x \in [0, 1]$ and f' is monotonic. Show that

$$\left| \int_0^1 e^{i\lambda f(x)} dx \right| \leq \frac{2}{\lambda}.$$

Here i is the imaginary unit.

Problem 4. Let $f : \mathbb{R}^3 \to \mathbb{R}$ with $f, \nabla f \in L^1(\mathbb{R}^3)$. Show that

$$\int_{\mathbb{R}^3} |f(x)|^{3/2} dx \leq \left(\int_{\mathbb{R}^3} |\nabla f(x)| dx \right)^{3/2}.$$

Problem 5. Fix a continuous function $f : [0,1] \to \mathbb{R}$. Consider the multiplication operator $M_f : C^0([0,1]) \to C^0([0,1])$ on the space $C^0([0,1])$ of continuous functions on $[0,1]$ defined by $(M_f g)(x) = f(x)g(x)$ for all $x \in [0,1]$ and $g \in C^0([0,1])$. Calculate $||M_f||$ and show that M_f is a compact operator if and only if $f \equiv 0$.

Problem 6.

Suppose that $f \in S(\mathbb{R})$, where $S(\mathbb{R})$ is the Schwartz space of infinitely differentiable rapidly decreasing functions

$$S(\mathbb{R}) = \{ f \in C^\infty(\mathbb{R}) : \sup_{x \in \mathbb{R}} |x^n f^{(m)}(x)| < \infty \}$$

for all nonnegative integers $n, m = 0, 1, 2, \ldots$.

Does

$$\int_{\mathbb{R}} f(x) x^n dx = 0, \quad n = 0, 1, 2, \ldots$$

imply that f is identically zero? Explain your answer. (Hint: use the Fourier transform).

Note: $f^{(m)}$ denotes the m-th derivative of f.

Spring 2014: PhD Analysis Preliminary Exam

Instructions:

1. *All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*

2. *Use separate sheets for the solution of each problem.*

Problem 1: Let (g_n) be a sequence of absolutely continuous functions on $[0,1]$ with $|g_n(0)| \leq 1$. Suppose also that for each n, $|g_n'(x)| \leq 1$ for Lebesgue almost everywhere $x \in [0,1]$. Show that there is a subsequence of (g_n) that converges uniformly to a Lipschitz function on $[0,1]$.

Problem 2: Let T be a linear operator from a Banach space X to a Hilbert space H. Show that T is bounded if and only if $x_n \rightharpoonup x$ implies that $T(x_n) \rightharpoonup T(x)$ for every weakly convergent sequence (x_n) in X.

Problem 3: Let $f, f_k : E \to [0, +\infty)$ be non-negative Lebesgue integrable functions on a measurable set $E \subseteq \mathbb{R}^n$. If (f_k) converges to f pointwise almost everywhere and

$$\int_E f_k dx \to \int_E f dx,$$

show that

$$\int_E |f - f_k| dx \to 0.$$

Problem 4: Let P_1 and P_2 be a pair of orthogonal projections onto H_1 and H_2, respectively, where H_1 and H_2 are closed subspaces of a Hilbert space H. Prove that $P_1 P_2$ is an orthogonal projection if and only if P_1 and P_2 commute. In that case, prove that $P_1 P_2$ is the orthogonal projection onto $H_1 \cap H_2$.

Problem 5: Let H be a (separable) Hilbert space with orthonormal basis $\{f_k\}_{k=1}^{\infty}$. Prove that the operator defined by

$$T(f_k) = \frac{1}{k} f_{k+1}, \quad k \geq 1,$$

is compact but has no eigenvalues.

Problem 6: Let $H_1 = L^2([-\pi, \pi])$ be the Hilbert space of functions $F(e^{i\theta})$ on the unit circle with inner product

$$(F, G) = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(e^{i\theta}) \overline{G(e^{i\theta})} d\theta.$$

Let H_2 be the space $L^2(\mathbb{R})$. Using the mapping

$$x \to \frac{i - x}{i + x}$$

of $\mathbb{R}$ to the unit circle, show that:
a) The correspondence $U : F \to f$, with

$$f(x) = \frac{1}{\pi^{1/2}(i+x)} F\left(\frac{i-x}{i+x}\right)$$

gives a unitary mapping of H_1 to H_2.
b) As a result,

$$\left\{\pi^{-1/2}\left(\frac{i-x}{i+x}\right)^n \frac{1}{i+x}\right\}_{n=-\infty}^{\infty}$$

is an orthonormal basis of $L^2(\mathbb{R})$.

Analysis Prelim Problems
Fall, 2014

Instructions:

1. *All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*

2. *Use separate sheets for the solution of each problem.*

Problem 1. Suppose $f : \mathbb{R} \to \mathbb{R}$ is twice continuously differentiable. Suppose $|f(x)| \leq 1$ and $|f''(x)| \leq 1$ for all $x \in \mathbb{R}$. Prove or disprove that $|f'(x)| \leq 2$ for all $x \in \mathbb{R}$.

Problem 2. Define the bounded linear operator $K : L^2(0,1) \to L^2(0,1)$ by

$$(Kf)(x) = \int_0^1 xy\,(1 - xy)\, f(y)\, dy.$$

Find the spectrum of K and classify it.

Problem 3. Let $\mathcal{H}$ be a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$, and let $\{e_n : n \in \mathbb{N}\}$ be an orthonormal basis of $\mathcal{H}$. Define a metric $d : B \times B \to \mathbb{R}$ on the closed unit ball B of $\mathcal{H}$ by

$$d(x, y) = \sum_{n=1}^{\infty} \frac{|\langle x - y, e_n \rangle|}{2^n}.$$

(a) Show that a sequence (x_k) in B converges to $x \in B$ with respect to the metric d if and only if it converges weakly to x in $\mathcal{H}$.

(b) Prove that (B, d) is a compact metric space.

Problem 4. Let $C_0(\mathbb{R})$ denote the Banach space of continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) \to 0$ as $|x| \to \infty$, equipped with the sup-norm.

(a) For $n \in \mathbb{N}$, define $f_n \in C_0(\mathbb{R})$ by

$$f_n(x) = \begin{cases} 1 & \text{if } |x| \leq n \\ n/|x| & \text{if } |x| > n \end{cases}$$

Show that $\mathcal{F} = \{f_n : n \in \mathbb{N}\}$ is a bounded, equicontinuous subset of $C_0(\mathbb{R})$, but that the sequence (f_n) has no uniformly convergent subsequence. Why doesn't this example contradict the Arzelà-Ascoli theorem?

(b) A family of functions $\mathcal{F} \subset C_0(\mathbb{R})$ is said to be *tight* if for every $\epsilon > 0$ there exists $R > 0$ such that

$$|f(x)| < \epsilon \qquad \text{for all } x \in \mathbb{R} \text{ with } |x| \geq R \text{ and all } f \in \mathcal{F}.$$

Prove that $\mathcal{F} \subset C_0(\mathbb{R})$ is precompact in $C_0(\mathbb{R})$ if it is bounded, equicontinuous, and tight.

Problem 5. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a smooth (C^∞) function with compact support. Prove that

$$\lim_{n\to\infty} \left\{ \sqrt{\frac{8}{\pi}} \int_0^\infty n \sin\left(n^2 x^2\right) f(x)\, dx \right\} = f(0).$$

Hint. You can use the fact that

$$\lim_{a\to\infty} \left\{ \int_0^a \sin\left(t^2\right)\, dt \right\} = \sqrt{\frac{\pi}{8}}.$$

Problem 6.

(a) By choosing a suitable even, periodic extension for f, calculate the Fourier cosine series for $f(x) = \sin x, \quad x \in [0, \pi]$.

(b) Deduce that

$$\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} = 1/2.$$

Spring 2013: PhD Analysis Preliminary Exam

Instructions:

1. *All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*
2. *Use separate sheets for the solution of each problem.*

Problem 1: Consider the Hilbert space $\mathcal{H} = L^2([0,1])$ with inner product $\langle f, h\rangle = \int\limits_0^1 f(x)h(x)\,dx$. Let

$$V = \{f \in L^2([0,1]) : \int_0^1 xf(x)dx = 0\} \subset \mathcal{H}$$

and $g(x) \equiv 1$. Find the closest element to g in V. Justify your answer.

Problem 2: Let $(B, \|\|)$ be a Banach space. Recall that the spectrum of a bounded linear operator $A \in \mathcal{L}(B)$ is defined as

$$\sigma(A) = \{\lambda \in \mathbb{C} : \lambda I - A \text{ is not invertible}\}.$$

Consider a sequence of bounded linear operators $A_n \in \mathcal{L}(B)$ which converges in norm to a bounded linear operator $A \in \mathcal{L}(B)$. Assume that all spectra are the same, i.e. $\sigma_0 := \sigma(A_1) = \sigma(A_2) = \dots$. Show that $\sigma_0 \subset \sigma(A)$.

Problem 3: Consider the function

$$f(x) = \begin{cases} 2\sin(x) + 3, & x > 0 \\ -2\sin(x) + c, & x \leq 0 \end{cases}.$$

Find its distributional derivative $\frac{df}{dx}$ and $\frac{d^2 f}{dx^2}$. For which values of c will $f \in W^{1,p}_{loc}(\mathbb{R})$? Justify your answer.

Problem 4: Prove that

$$\lim_{\epsilon \to 0^+} \int_0^\infty \frac{\epsilon}{\epsilon^2 + x} \sin(1/x)dx = 0\,.$$

Problem 5: Prove that the image of the space $C^k(\mathbb{T})$ of k times continuously differentiable functions on the unit circle under the Fourier transform is contained in the set of sequences satisfying $|c_n| = o(|n|^{-k})$ and contains the set of sequences satisfying $|c_n| = o(|n|^{-k-1-\epsilon}),\ \epsilon > 0$.
(Recall: $f(n) = o(h(n))$ as $n \to \infty$ means that for every $\delta > 0$ there exists an N such that $|f(n) \leq \delta|g(n)|$ for all $n > N$).

Problem 6: Let I be the interval $(0,1)$ and $q \geq p \geq 1$. Show that there exists a constant $C = C\,(p,q,I)$ such that

$$\|u\|_{L^q(I)} \leq C\,\|u\|_{W^{1,p}(I)}$$

for all $u \in W_0^{1,p}(I)$. (Hint: First show that $\|u\|_{L^\infty(I)} \leq C\,\|u\|_{W^{1,p}(I)}$ for all $u \in W_0^{1,p}(I)$.)

Fall 2013: PhD Analysis Preliminary Exam

Instructions:

1. *All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*
2. *Use separate sheets for the solution of each problem.*

Problem 1: Find $\inf \int_0^1 |f(x) - x|^2 dx$ where the infimum is taken over all $f \in L^2([0,1])$ such that $\int_0^1 f(x)(x^2-1)dx = 1$.

Problem 2: Let $L^2([0,1])$ denote the Hilbert space of complex valued square integrable functions on $[0,1]$ with the usual inner product

$$(f,g) = \int_0^1 f(x)\overline{g(x)}dx.$$

Define $T : L^2([0,1]) \to L^2([0,1])$ by

$$(Tf)(x) = \int_0^x f(t)dt, \text{ for } x \in [0,1].$$

(a) Show that T is bounded.

(b) Show that T has no eigenvalues.

(c) Find $\lim_{n\to\infty} ||T^n||$.

Problem 3: For $\delta > 0$ small, let $u \in L^{\frac{3}{2}+\delta}(\mathbb{R}^3) \cap L^{\frac{3}{2}-\delta}(\mathbb{R}^3)$. Prove that $v = u * \frac{1}{|x|} \in L^\infty(\mathbb{R}^3)$ and provide a bound for $\|v\|_{L^\infty(\mathbb{R}^3)}$ which depends only on $\|u\|_{L^{\frac{3}{2}+\delta}(\mathbb{R}^3)}$.

Problem 4: Let H be a separable infinite dimensional Hilbert space and suppose that $e_1, e_2, \ldots$ is an orthonormal system in H. Let $f_1, f_2, \ldots$ be another orthonormal system which is complete (i.e. the closure of the span of $\{f_i\}_i$ is all of H.) Prove that if $\sum_{n=1}^\infty ||e_n - f_n||^2 < 1$ then $\{e_i\}_i$ is also a complete orthonormal system.

Problem 5: Suppose A is a compact operator on an infinite dimensional Hilbert space $\mathcal{H}$. Show that A does not have a bounded inverse operator.

Problem 6: Let $\mathcal{S}(\mathbb{R}^n)$ be the Schwarz space. Show that $\mathcal{S}(\mathbb{R}^n) \subseteq L^p(\mathbb{R}^n)$ for any $1 \le p \le \infty$.

Spring 2012: PhD Analysis Preliminary Exam

Instructions:

1. *All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*

2. *Use separate sheets for the solution of each problem.*

Problem 1:
For $u \in L^1(0,\infty)$, consider the integral

$$v(x) = \int_0^\infty \frac{u(y)}{x+y} dy$$

defined for $x > 0$. Show that $v(x)$ is infinitely differentiable away from the origin. Prove that $v' \in L^1(\epsilon, \infty)$ for any $\epsilon > 0$. Explain what happens in the limit as $\epsilon \to 0$.

Problem 2. Let $X \subset L^2(0, 2\pi)$ be the set of all functions $u(x)$ such that

$$u(x) = \lim_{K\to\infty} \sum_{k=-K}^{K} a_k e^{ikx} \text{ in } L^2\text{-norm, with } |a_k| \le (1+|k|)^{-1}.$$

Prove that X is compact in $L^2(0, 2\pi)$.

Problem 3. For $\epsilon > 0$, we set

$$\eta_\epsilon(x) = \frac{1}{\pi} \sin\left(\frac{\epsilon\pi x}{x^2+\epsilon^2}\right) \frac{\epsilon}{x^2+\epsilon^2},$$

and define the convolution for $u \in L^2(\mathbb{R})$:

$$\eta_\epsilon * u\,(x) = \int_{\mathbb{R}} \eta_\epsilon(x-y) u(y) dy.$$

For $\epsilon > 0$, prove that $\sqrt{\epsilon}(\eta_\epsilon * u)(x)$ is bounded as a function of x and ϵ, and that $\eta_\epsilon * u$ converges strongly in $L^2(\mathbb{R})$ as $\epsilon \to 0$. What is the limit?

Problem 4. Let $u_n : [0,1] \longrightarrow [0,\infty)$ denote a sequence of measurable functions satisfying

$$\sup_n \int_0^1 u_n(x) \log(2 + u_n(x))\, dx < \infty\,.$$

If $u_n(x) \longrightarrow u(x)$ almost everywhere, show that $u \in L^1(0,1)$ and that $u_n \longrightarrow u$ in L^1 strongly. (**Hint.** One possible strategy is Egoroff's Theorem.)

Problem 5. Let $u : [0,1] \longrightarrow \mathbb{R}$ be absolutely continuous, satisfy $u(0) = 0$, and

$$\int_0^1 |u'(x)|^2 dx < \infty\,.$$

Prove that

$$\lim_{x \to 0^+} \frac{u(x)}{x^{\frac{1}{2}}}$$

exists and determine the value of this limit.

Problem 6. Consider on $\mathbb{R}^2$ the distribution defined by the locally integrable function

$$E(x,t) = \begin{cases} \frac{1}{2} & \text{if } t - |x| > 0 \\ 0 & \text{if } t - |x| < 0 \end{cases}\,.$$

Compute the distributional derivative

$$\frac{\partial^2 E}{\partial t^2} - \frac{\partial^2 E}{\partial x^2}\,.$$

Fall 2012: PhD Analysis Preliminary Exam

Instructions:

1. *All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*
2. *Use separate sheets for the solution of each problem.*

Problem 1:
Show that the space of all continuous functions on the interval $[0, 1]$ with the sup norm $||f|| = \max |f(x)|$ is not a Hilbert space.

Problem 2. Suppose φ is a real-valued continuous function on the interval $[0, 1]$, and T is a linear operator on $L^2 [0, 1]$ given by

$$(Tf)(x) = \varphi(x) \int_0^1 \varphi(t) f(t)\, dt$$

for all $f \in L^2 [0, 1]$. Show that

(a) T is self-adjoint.

(b) there exists a number $\lambda \geq 0$ such that $T^2 = \lambda T$.

(c) Find the spectral radius $r(T)$ of T.

Problem 3. Let T be a bounded linear operator on a Hilbert space $\mathcal{H}$. Show that

(a) If $\|T\| \leq 1$, then T and its adjoint operator T^* have the same fixed point. i.e. Show that for $x \in \mathcal{H}$,

$$Tx = x \iff T^* x = x.$$

(b) Let λ be an eigenvalue of T. Is it true that its complex conjugate $\bar{\lambda}$ must be an eigenvalue of T^*? Is it true that $\bar{\lambda}$ must be in the spectrum of T^*? Justify your answers.

Problem 4. The heat kernel on $\mathbb{R}^3$ is given by $H_t(x) = (4\pi t)^{-3/2} e^{-|x|^2/(4t)}$ where $|x|$ denotes the Euclidean norm of $x \in \mathbb{R}^3$. Prove that if $u \in L^3(\mathbb{R}^3)$, then $t^{1/2}\|H_t * u\|_{L^\infty(\mathbb{R}^3)} \to 0$ as $t \to 0^+$. (Note that $*$ denotes convolution.)

Problem 5. Let M be a bounded subset of $C\,[a,b]$ with the sup norm and

$$A = \left\{ F\,(x) = \int_a^x f\,(t)\,dt : f \in M \right\}.$$

Show that A is a precompact subset of $C\,[a,b]$.

Problem 6. Consider the one-dimensional function f. Prove that if

$$\int_{-\infty}^{\infty} |\hat{f}(k)|^2 (1 + |k|^2)^s dk < \infty$$

for some $s > 3/2$ then f is globally Lipschitz, i.e., there exists a constant K such that $|f(x) - f(y)| \leq K|x - y|$.

Spring 2011: PhD Analysis Preliminary Exam

Instructions:

1. *All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*

2. *Use separate sheets for the solution of each problem.*

Problem 1:
Let $\Omega = (0,1)$, the open unit interval in $\mathbb{R}$, and consider the sequence of functions $f_n(x) = ne^{-nx}$. Prove that $f_n \not\rightharpoonup f$ weakly in $L^1(\Omega)$, i.e., the sequence f_n does not converge in the weak topology of $L^1(\Omega)$.
(**Hint**: Prove by contradiction.)

Problem 2:
Let $\Omega = (0,1)$, and consider the linear operator $A = -\frac{d^2}{dx^2}$ acting on the Sobolev space of functions X where

$$X = \{u \in H^2(\Omega) \mid u(0) = 0\,, u(1) = 0\}\,,$$

and where

$$H^2(\Omega) = \{u \in L^2(\Omega) \mid \frac{du}{dx} \in L^2(\Omega)\,, \frac{d^2u}{dx^2} \in L^2(\Omega)\}\,.$$

Find all of the eigenfunctions of A belonging to the linear span of

$$\{\cos(\alpha x), \sin(\alpha x) \mid \alpha \in \mathbb{R}\}\,,$$

as well as their corresponding eigenvalues.

Problem 3:
Let $\Omega = (0,1)$, the open unit interval in $\mathbb{R}$, and set

$$v(x) = (1 + |\log x|)^{-1}.$$

Show that $v \in W^{1,1}(\Omega)$ and that $v(0) = 0$, but that $\frac{v}{x} \notin L^1(\Omega)$. (This shows the failure of Hardy's inequality in L^1.) Note, that $W^{1,1}(\Omega) = \{u \in L^1(\Omega) \mid \frac{du}{dx} \in L^1(\Omega)\}$, where $\frac{du}{dx}$ denotes the weak derivative.

Problem 4:
Let $f(x)$ be a periodic continuous function on $\mathbb{R}$ with period 2π. Show that

$$\hat{f}(\xi) = \sum_{n=-\infty}^{\infty} b_n \tau_n \delta \text{ in } \mathcal{D}' , \tag{1}$$

that is, that equality in equation (1) holds in the sense of distributions, and relate b_n to the coefficients of the Fourier series. Note that δ denotes the Dirac distribution and τ_y is the translation operator, given by $\tau_y f(x) = f(x+y)$.
(**Hint:** Write $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$ with convergence in $L^2(0,2\pi)$ and where the coefficients $c_n = \frac{1}{2\pi}\int_0^{2\pi} e^{-inx} f(x)dx$.)

Problem 5:
Let $f(x)$ be a periodic continuous function on $\mathbb{R}$ with period 2π. Given $\epsilon > 0$, prove that for $N < \infty$ there is a finite Fourier series

$$\phi(x) = a_0 + \sum_{n=1}^{N} [a_n \cos(nx) + b_n \sin(nx)] \tag{2}$$

such that

$$|\phi(x) - f(x)| < \epsilon \quad \forall x \in \mathbb{R} .$$

This shows that the space of real-valued trigonometric polynomials on $\mathbb{R}$ (functions which can be expressed as in (2)) are *uniformly* dense in the space of periodic continuous function on $\mathbb{R}$ with period 2π.
(**Hint:** The Stone-Weierstrass theorem states that if X is compact in $\mathbb{R}^d$, $d \in \mathbb{N}$, then the algebra of all real-valued polynomials on X (with coordinates $(x_1, x_2, ..., x_d)$) is dense in $C(X)$.)

Problem 6:
For $\alpha \in (0,1]$, the space of Hölder continuous functions on the interval $[0,1]$ is defined as

$$C^{0,\alpha}([0,1]) = \{u \in C([0,1]) \ : \ |u(x) - u(y)| \leq C|x-y|^{\alpha} \, , x, y \in [0,1]\} ,$$

and is a Banach space when endowed with the norm

$$\|u\|_{C^{0,\alpha}([0,1])} = \sup_{x\in[0,1]} |u(x)| + \sup_{x,y\in[0,1]} \frac{|u(x)-u(y)|}{|x-y|^{\alpha}} .$$

Prove that the closed unit ball $\{u \in C^{0,\alpha}([0,1]) \ : \ \|u\|_{C^{0,\alpha}([0,1])} \leq 1\}$ is a compact set in $C([0,1])$.
(**Hint:** The Arzela-Ascoli theorem states that if a family of continuous functions U is equicontinuous and uniformly bounded on $[0,1]$, then each sequence u_n in U has a uniformly convergent subsequence. Recall that U is uniformly bounded on $[0,1]$ if there exists $M > 0$ such that $|u(x)| < M$ for all $x \in [0,1]$ and all $u \in U$. Further, recall that U is equicontinuous at $x \in [0,1]$ if given any $\epsilon > 0$, there exists $\delta > 0$ such that $|u(x) - u(y)| < \epsilon$ for all $|x - y| < \delta$ and every $u \in U$.)

Fall 2011: PhD Analysis Preliminary Exam

Instructions:

1. *All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*

2. *Use separate sheets for the solution of each problem.*

Problem 1:
Let (X, d) be a metric space and let (x_n) be a sequence in X. For the purpose of this problem adopt the following definition: $x \in X$ is called a *cluster point* of (x_n) iff there exists a subsequence $(x_{n_k})_{k \geq 0}$ such that $\lim_k x_{n_k} = x$.

(a) Let $(a_n)_{n \geq 0}$ be a sequence of distinct points in X. Construct a sequence $(x_n)_{n \geq 0}$ in X such that for all $k = 0, 1, 2, \ldots$, a_k is a cluster point of (x_n).

(b) Can a sequence (x_n) in a metric space have an *uncountable* number of cluster points? Prove your answer. (If you answer yes, give an example with proof. If you answer no, prove that such a sequence cannot exists). You may use without proof that $\mathbb{Q}$ is countable and $\mathbb{R}$ is uncountable.

Problem 2:
Let X be a real Banach space and X^* its Banach space dual. For any bounded linear operator $T \in \mathcal{B}(X)$, and $\phi \in X^*$, define the functional $T^*\phi$ by

$$T^*\phi(x) = \phi(Tx), \text{ for all } x \in X.$$

(a) Prove that T^* is a bounded operator on X^* with $\|T^*\| \leq \|T\|$.

(b) Suppose $0 \neq \lambda \in \mathbb{R}$ is an eigenvalue of T. Prove that λ is also an eigenvalue of T^*. (Hint 1: first prove the result for $\lambda = 1$. Hint 2: For $\phi \in X^*$, consider the sequence of Cesàro means $\psi_N = N^{-1} \sum_{n=1}^{N} \phi_n$, of the sequence ϕ_n defined by $\phi_n(x) = \phi(T^n x)$.)

Problem 3:
Let $\mathcal{H}$ be a complex Hilbert space and denote by $\mathcal{B}(\mathcal{H})$ the Banach space of all bounded linear transformations (operators) of $\mathcal{H}$ considered with the operator norm.

(a) What does it mean for $A \in \mathcal{B}(\mathcal{H})$ to be *compact*? Give a definition of compactness of an operator A in terms of properties of the image of bounded sets, e.g., the set $\{Ax \mid x \in \mathcal{H}, \|x\| \leq 1\}$.

(b) Suppose $\mathcal{H}$ is separable and let $\{e_n\}_{n\geq 0}$ be an orthonormal basis of $\mathcal{H}$. For $n \geq 0$, let P_n denote the orthogonal projection onto the subspace spanned by $e_0, \ldots, e_n$. Prove that $A \in \mathcal{B}(\mathcal{H})$ is compact iff the sequence $(P_n A)_{n\geq 0}$ converges to A in norm.

Problem 4:
Let $\Omega \subset \mathbb{R}^n$ be open, bounded, and smooth. Suppose that $\{f_j\}_{j=1}^{\infty} \subset L^2(\Omega)$ and $f_j \rightharpoonup g_1$ weakly in $L^2(\Omega)$ and that $f_j(x) \to g_2(x)$ a.e. in Ω. Show that $g_1 = g_2$ a.e. (**Hint:** Use Egoroff's theorem which states that given our assumptions, for all $\epsilon > 0$, there exists $E \subset \Omega$ such that $\lambda(E) < \epsilon$ and $f_j \to g_2$ uniformly on E^c.)

Problem 5:
Let $u(x) = (1 + |\log x|)^{-1}$. Prove that $u \in W^{1,1}(0,1)$, $u(0) = 0$, but $\frac{u}{x} \notin L^1(0,1)$.

Problem 6:
Let $H = \{f \in L^2(0, 2\pi) \ : \ \int_0^{2\pi} f(x)dx = 0\}$. We define the operator Λ as follows:

$$(\Lambda f)(x) = \int_0^x f(y)dy \,.$$

(a) Prove that $\Lambda : H \to L^2(0, 2\pi)$ is continuous.

(b) Use the Fourier series to show that the following estimate holds:

$$\|\Lambda f\|_{H_0^1(0,2\pi)} \leq C \|f\|_{L^2(0,2\pi)} \,,$$

where C denotes a constant which depends only on the domain $(0, 2\pi)$. (Recall that $\|u\|^2_{H_0^1(0,2\pi)} = \int_0^{2\pi} |\frac{du}{dx}(x)|^2 dx$.)

Spring 2010: PhD Analysis Preliminary Exam

Instructions:

1. *All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*
2. *Use separate sheets for the solution of each problem.*

Problem 1:
Let (X,d) be a complete metric space, $\bar{x} \in X$ and $r > 0$. Set $D := \{x \in X : d(x,\bar{x}) \leq r\}$, and let $f : D \to X$ satisfying

$$d(f(x), f(y)) \leq k\, d(x,y)$$

for any $x, y \in D$, where $k \in (0,1)$ is a constant.
Prove that if $d(\bar{x}, f(\bar{x})) \leq r(1-k)$ then f admits a unique fixed point. (Guidelines: Assume the Banach fixed point theorem, also known as the contraction mapping theorem.)

Problem 2:
Give an example of two normed vector spaces, X and Y, and of a sequence of operators, $\{T_n\}_{n=0}^{\infty}$, $T_n \in L(X,Y)$ ($L(X,Y)$ is the space of the continuous operators from X to Y, with the topology induced by the operator norm) such that $\{T_n\}_{n=0}^{\infty}$ is a Cauchy sequence but it does not converge in $L(X,Y)$. (Notice that Y cannot be a Banach space otherwise $L(X,Y)$ is complete.)

Problem 3:
Let (a_n) be a sequence of positive numbers such that

$$\sum_{n=1}^{\infty} a_n^3$$

converges. Show that

$$\sum_{n=1}^{\infty} \frac{a_n}{n}$$

also converges.

Problem 4: Suppose that $h : [0,1]^2 \to [0,1]^2$ is a continuously differentiable function from the square to the square with a continuously differentiable inverse h^{-1}. Define an operator T on the Hilbert space $L^2([0,1]^2)$ by the formula $T(f) = f \circ h$. Prove that T is a well-defined bounded operator on this Hilbert space.

Problem 5: Let $H^s(\mathbb{R})$ denote the Sobolev space of order s on the real line $\mathbb{R}$, and let

$$\|u\|_s = \left(\int_{\mathbb{R}} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi \right)^{\frac{1}{2}}$$

denote the norm on $H^s(\mathbb{R})$, where $\hat{u}(\xi) = \frac{1}{2\pi} \int_{\mathbb{R}} u(x) e^{-ix\xi} dx$ denotes the Fourier transform of u.

Suppose that $r < s < t$, all real, and $\epsilon > 0$ is given. Show that there exists a constant $C > 0$ such that

$$\|u\|_s \leq \epsilon \|u\|_t + C \|u\|_r \quad \forall u \in H^t(\mathbb{R}) .$$

Problem 6: Let $f : [0,1] \to \mathbb{R}$. Show that f is continuous if and only if the graph of f is compact in $\mathbb{R}^2$.

ANALYSIS

Problem 1. Let $f(x, y)$ denote a C^1 function on $\mathbb{R}^2$. Suppose that

$$f(0,0) = 0.$$

Prove that there exist two functions, $A(x, y)$ and $B(x, y)$, both continuous on $\mathbb{R}^2$ such that

$$f(x, y) = xA(x, y) + yB(x, y) \quad \forall (x, y) \in \mathbb{R}^2$$

(**Hint**: Consider the function $g(t) = f(tx, ty)$ and express $f(x, y)$ in terms of g via the fundamental theorem of calculus .)

Problem 2. The Fourier transform $\mathcal{F}$ of a distribution is defined via the duality relation

$$\langle \mathcal{F} f, \phi \rangle = \langle f, \mathcal{F}^* \phi \rangle$$

for all $\phi \in C_0^\infty(\mathbb{R})$, the smooth compactly-supported test functions on $\mathbb{R}$, where

$$\mathcal{F}^* \phi(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{i\xi x} \phi(\xi) d\xi \, .$$

Explicitly compute $\mathcal{F} f$ for the function

$$f(x) = \begin{cases} x, & x > 0 \\ 0, & x \leq 0 \end{cases} .$$

Problem 3. Let $\{P_n(x)\}_{n=1}^\infty$ denote a sequence of polynomials on $\mathbb{R}$ such that

$$P_n \to 0 \text{ uniformly on } \mathbb{R} \text{ as } n \to \infty.$$

Prove that, for n sufficiently large, all P_n are constant polynomials.

Problem 4. For $g \in L^1(\mathbb{R}^3)$, the convolution operator G is defined on $L^2(\mathbb{R}^3)$ by

$$Gf(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} g(x - y) f(y) dy \, , \quad f \in L^2(\mathbb{R}^3) \, .$$

Prove that the operator G with

$$g(x) = \frac{1}{4\pi} \frac{e^{-|x|}}{|x|} \, , \quad x \in \mathbb{R}^3,$$

is a bounded operator on $L^2(\mathbb{R}^3)$, and the operator norm $\|G\|_{op} \leq 1$.

Problem 5. Consider the map which associates to each sequence $\{x_n \, : \, n \in \mathbb{N} \, , \, x_n \in \mathbb{R}\}$ the sequence, $\{(F(\{x_n\}))_m \, ; m \in \mathbb{N} \, , \, (F(\{x_n\}))_m \in \mathbb{R}\}$, defined as follows:

$$\Big\{ F(\{x_n\}) \Big\}_m := \frac{x_m}{m} \quad \text{for} \quad m = 1, 2, \ldots$$

(1) Determine (with proof) the values of $p \in [1, \infty]$ for which the map $F : l^p \to l^1$ is well-defined and continuous.

(2) Next, determine the values of $q \in [1, \infty]$ for which the map $F : l^q \to l^2$ is well-defined and continuous.

Note for $1 \leq p < \infty$, l^p denotes the space of sequences $\{x_n\}_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} |x_n|^p < \infty$, while l^{∞} denotes the space of sequences $\{x_n\}_{n=1}^{\infty}$ such that $\sup_{n \in \mathbb{N}} |x_n| < \infty$.

Problem 6. For each of the following, determine if the statement is true (always) or false (not always true). If true, give a brief proof, e.g. by citing a relevant theorem; if false, give a counterexample.

Let $\mathbb{H}$ denote a separable Hilbert space and (x_n) a sequence of $\mathbb{H}$.

(a) If (x_n) is weakly convergent then it is strongly convergent.

(b) If (x_n) is strongly convergent then it is bounded.

(c) If (x_n) is weakly convergent then it is bounded.

(d) If (x_n) is bounded, there exists a strongly convergent subsequence of (x_n).

(e) If (x_n) is bounded, there exists a weakly convergent subsequence of (x_n).

(f) If (x_n) is weakly convergent and T is a bounded linear operator from $\mathbb{H}$ to $\mathbb{R}^d$, for some d, then $T(x_n)$ converges in $\mathbb{R}^d$.

Winter 2009: PhD Analysis Preliminary Exam

Instructions:

1. *All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*

2. *Use separate sheets for the solution of each problem.*

Problem 1: Let $1 < p < 2$.
(a) Give an example of a function $f \in L^1(\mathbb{R})$ such that $f \notin L^p(\mathbb{R})$ and a function $g \in L^2(\mathbb{R})$ such that $g \notin L^p(\mathbb{R})$.
(b) If $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, prove that $f \in L^p(\mathbb{R})$
Problem 2:
(a) State the Weierstrass approximation theorem.
(b) Suppose that $f : [0,1] \longrightarrow \mathbb{R}$ is continuous and

$$\int_0^1 x^n f(x)\,dx = 0$$

for all non-negative integers n. Prove that $f = 0$.
Problem 3:
(a) Define strong convergence, $x_n \longrightarrow x$, and weak convergence, $x_n \rightharpoonup x$, of a sequence (x_n) in a Hilbert space $\mathcal{H}$.
(b) If $x_n \rightharpoonup x$ weakly in $\mathcal{H}$ and $\|x_n\| \to \|x\|$, prove that $x_n \to x$ strongly.
(c) Give an example of a Hilbert space $\mathcal{H}$ and sequence (x_n) in $\mathcal{H}$ such that $x_n \rightharpoonup x$ weakly and

$$\|x\| < \liminf_{n\to\infty} \|x_n\|.$$

Problem 4: Suppose that $T : \mathcal{H} \to \mathcal{H}$ is a bounded linear operator on a complex Hilbert space $\mathcal{H}$ such that

$$T^* = -T, \qquad T^2 = -I$$

and $T \neq \pm iI$. Define

$$P = \frac{1}{2}(I + iT), \qquad Q = \frac{1}{2}(I - iT).$$

(a) Prove that P, Q are orthogonal projections on $\mathcal{H}$.

(b) Determine the spectrum of T, and classify it.

Problem 5: Let $\mathcal{S}(\mathbb{R})$ be the Schwartz space of smooth, rapidly decreasing functions $f : \mathbb{R} \to \mathbb{C}$. Define an operator $H : \mathcal{S}(\mathbb{R}) \to L^2(\mathbb{R})$ by

$$\widehat{(Hf)}(\xi) = i\mathrm{sgn}(\xi)\hat{f}(\xi) = \begin{cases} i\hat{f}(\xi) & \text{if } \xi > 0, \\ -i\hat{f}(\xi) & \text{if } \xi < 0, \end{cases}$$

where $\hat{f}$ denotes the Fourier transform of f.

(a) Why is $Hf \in L^2(\mathbb{R})$ for any $f \in \mathcal{S}(\mathbb{R})$?

(b) If $f \in \mathcal{S}(\mathbb{R})$ and $Hf \in L^1(\mathbb{R})$, show that

$$\int_{\mathbb{R}} f(x)\, dx = 0.$$

(Hint: you may want to use the Riemann-Lebesgue Lemma)

Problem 6: Let Δ denote the Laplace operator in $\mathbb{R}^3$.

(a) Prove that

$$\lim_{\epsilon \to 0} \int_{B_\epsilon^c} \frac{1}{|\mathbf{x}|} \Delta f(\mathbf{x}) d\mathbf{x} = 4\pi f(0), \quad \forall f \in \mathcal{S}(\mathbb{R}^3)$$

where B_ϵ^c is the complement of the ball of radius ϵ centered at the origin.

(b) Find the solution u of the Poisson problem

$$\Delta u = 4\pi f(\mathbf{x}), \quad \lim_{|\mathbf{x}| \to \infty} u(\mathbf{x}) = 0$$

for $f \in \mathcal{S}(\mathbb{R}^3)$.

Fall 2009: PhD Analysis Preliminary Exam

Instructions:

1. *All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*

2. *Use separate sheets for the solution of each problem.*

Problem 1. For $\epsilon > 0$, let η_ϵ denote the family of *standard* mollifiers on $\mathbb{R}^2$. Given $u \in L^2(\mathbb{R}^2)$, define the functions

$$u_\epsilon = \eta_\epsilon * u \text{ in } \mathbb{R}^2 .$$

Prove that

$$\epsilon \|Du_\epsilon\|_{L^2(\mathbb{R}^2)} \leq \|u\|_{L^2(\mathbb{R}^2)} ,$$

where the constant C depends on the mollifying function, but not on u.

Problem 2.
Let $B(0,1) \subset \mathbb{R}^3$ denote the unit ball $\{|x| < 1\}$. Prove that $\log|x| \in H^1(B(0,1))$.

Problem 3. Prove that the continuous functions of compact support are a dense subspace of $L^2(\mathbb{R}^d)$.

Problem 4. There are several senses in which a sequence of bounded operators $\{T_n\}$ can converge to a bounded operator T (in a Hilbert space $\mathcal{H}$). First, there is convergence in the norm, that is, $\|T_n - T\| \to 0$, as $n \to \infty$. Next, there is a weaker convergence, which happens to be called strong convergence, that requires that $T_n f \to Tf$, as $n \to \infty$, for every vector $f \in \mathcal{H}$. Finally, there is weak convergence that requires $(T_n f, g) \to (Tf, g)$ for every pair of vectors $f, g \in \mathcal{H}$.

(a) Show by examples that weak convergence does not imply strong convergence, nor does strong convergence imply convergence in norm.

(b) Show that for any bounded operator T there is a sequence $\{T_n\}$ of bounded operators of finite rank so that $T_n \to T$ strongly as $n \to \infty$.

Problem 5. Let $\mathcal{H}$ be a Hilbert space. Prove the following variants of the spectral theorem.

(a) If T_1 and T_2 are two linear symmetric and compact operators on $\mathcal{H}$ that commute (that is, $T_1T_2 = T_2T_1$), show that they can be diagonalized sim ultaneously. In other words, there exists an orthonormal basis for $\mathcal{H}$ which consists of eigenvectors for both T_1 and T_2.

(b) A linear operator on $\mathcal{H}$ is *normal* if $TT^* = T^*T$. Prove that if T is normal and compact, then T can be diagonalized.

(c) If U is unitary, and $U = \lambda I - T$, where T is compact, then U can be diagonalized.

Problem 6. Prove that a normed linear space is complete if and only if every absolutely summable sequence is summable.

Fall 2007: PhD Analysis Preliminary Exam

Instructions:

1. *All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*

2. *Use separate sheets for the solution of each problem.*

Problem 1: Suppose that $f : [0, 1] \to \mathbb{R}$ is continuous. Prove that

$$\lim_{n\to\infty} \int_0^1 f(x^n)\, dx$$

exists and evaluate the limit. Does the limit always exist if f is only assumed to be integrable?

Problem 2: Suppose that for each $n \in \mathbb{Z}$, we are given a real number ω_n. For each $t \in \mathbb{R}$, define a linear operator $T(t)$ on 2π-periodic functions by

$$T(t)\left(\sum_{n\in\mathbb{Z}} f_n e^{inx}\right) = \sum_{n\in\mathbb{Z}} e^{i\omega_n t} f_n e^{inx},$$

where $f(x) = \sum_{n\in\mathbb{Z}} f_n e^{inx}$ with $f_n \in \mathbb{C}$.

(a) Show that $T(t) : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ is a unitary map.
(b) Show that $T(s)T(t) = T(s + t)$ for all $s, t \in \mathbb{R}$.
(c) Prove that if $f \in C^\infty(\mathbb{T})$, meaning that it has continuous derivatives of all orders, then $T(t)f \in C^\infty(\mathbb{T})$.

Problem 3: Let $(X, \|\cdot\|_X), (Y, \|\cdot\|_Y), (Z, \|\cdot\|_Z)$ be Banach spaces, with X compactly embedded in Y, and Y continuously embedded in Z (meaning that: $X \subset Y \subset Z$; bounded sets in $(X, \|\cdot\|_X)$ are precompact in $(Y, \|\cdot\|_Y)$; and there is a constant M such that $\|x\|_Z \le M\|x\|_Y$ for every $x \in Y$). Prove that for every $\varepsilon > 0$ there exists a constant $C(\varepsilon)$ such that

$$\|x\|_Y \le \varepsilon\|x\|_X + C(\varepsilon)\|x\|_Z \qquad \text{for every } x \in X.$$

Problem 4: Let $\mathcal{H}$ be the weighted L^2-space

$$\mathcal{H} = \left\{ f : \mathbb{R} \to \mathbb{C} \,\middle|\, \int_{\mathbb{R}} e^{-|x|} |f(x)|^2 \, dx < \infty \right\}$$

with inner product

$$\langle f, g \rangle = \int_{\mathbb{R}} e^{-|x|} \overline{f(x)} g(x) \, dx.$$

Let $T : \mathcal{H} \to \mathcal{H}$ be the translation operator

$$(Tf)(x) = f(x+1).$$

Compute the adjoint T^* and the operator norm $\|T\|$.

Problem 5: (a) State the Rellich Compactness Theorem for the space $W^{1,p}(\Omega)$ for $\Omega \subset \mathbb{R}^n$. Recall that the Sobolev conjugate exponent is defined as $p^* = \frac{np}{n-p}$, and that there are some constraints on the set Ω.
(b) Suppose that $\{f_n\}_{n=1}^{\infty}$ is a bounded sequence in $H^1(\Omega)$ for $\Omega \subset \mathbb{R}^3$ open, bounded, and smooth. Show that there exists an $f \in H^1(\Omega)$ such that for a subsequence $\{f_{n_\ell}\}_{\ell=1}^{\infty}$,

$$f_{n_\ell} D f_{n_\ell} \rightharpoonup f \, Df \quad \text{weakly in} \quad L^2(\Omega),$$

where $D = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right)$ denotes the (weak) gradient operator.

Problem 6: Let $\Omega := B\left(0, \frac{1}{2}\right) \subset \mathbb{R}^2$ denote the open ball of radius $\frac{1}{2}$. For $x = (x_1, x_2) \in \Omega$, let

$$u(x_1, x_2) = x_1 x_2 \left[\log\left(|\log(|x|)| \right) - \log\log 2 \right] \quad \text{where } |x| = \sqrt{x_1^2 + x_2^2}\,.$$

(a) Show that $u \in C^1(\bar{\Omega})$.

(b) Show that $\frac{\partial^2 u}{\partial x_j^2} \in C(\bar{\Omega})$ for $j = 1, 2$, but that $u \notin C^2(\bar{\Omega})$.

(c) Using the elliptic regularity theorem for the Dirichlet problem on the disc, show that $u \in H^2(\Omega)$.